A NOTE ON BAZILEVIČ FUNCTIONS

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ABSTRACT. We provide a few sufficient conditions for a normalized analytic function in the unit disk to be a Bazilevič function of prescribed type.

1. Introduction

Throughout the paper, $\mathscr{A}$ denotes the class of analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized so that $f(0) = 0$ and $f'(0) = 1$.

Let α and β be real numbers with $\alpha > 0$ and set $\gamma = \alpha + \mathrm{i}\beta$. An $f \in \mathscr{A}$ is called a *Bazilevič* function of type (α, β) if

$$f(z) = \left[\gamma \int_0^z g(\zeta)^\alpha h(\zeta)\zeta^{\mathrm{i}\beta-1}\mathrm{d}\zeta\right]^{1/\gamma} = z\left[\gamma \int_0^1 \left(\frac{g(tz)}{tz}\right)^\alpha h(tz)t^{\gamma-1}\mathrm{d}t\right]^{1/\gamma} \tag{1.1}$$

for a starlike (univalent) function g in $\mathscr{A}$ and an analytic function h with $h(0) = 1$ satisfying $\mathrm{Re}\,(\mathrm{e}^{\mathrm{i}\lambda}h) > 0$ in $\mathbb{D}$ for some $\lambda \in \mathbb{R}$. Here and hereafter, $t^{\gamma-1} = \mathrm{e}^{(\gamma-1)\log t}$ with $\log t \in (-\infty, 0)$ for $0 < t < 1$ and, for a non-vanishing analytic function G with $G(0) = 1$, any power G^δ, $\delta \in \mathbb{C}$, will be understood as $\exp(\delta \log G)$, where $\log G$ means the analytic branch in $\mathbb{D}$ with $\log G(0) = 0$. We denote by $\mathscr{B}(\alpha, \beta)$ the class of Bazilevič functions of type (α, β). If we specify the real number λ in the above definition, we denote by $\mathscr{B}_\lambda(\alpha, \beta)$ the corresponding subclass of $\mathscr{B}(\alpha, \beta)$.

Let $\mathscr{S}, \mathscr{S}^*, \mathscr{K}, \mathscr{C}$, and $\mathscr{S}_{\mathrm{p}}(\lambda), -\pi/2 < \lambda < \pi/2$, denote the subclasses of $\mathscr{A}$ of functions univalent, starlike, convex, close-to-convex, and λ-spirallike, respectively. (For these classes, see [3] for instance, though the notation is not same as here.) It is well known that the inclusion relations $\mathscr{K} \subset \mathscr{S}^* \subset \mathscr{C} \subset \mathscr{S}$ are valid. For $\lambda \in \mathbb{R}$, we also denote by $\mathscr{P}_\lambda$ the class of analytic functions h with $h(0) = 1$ and $\mathrm{Re}\,(\mathrm{e}^{\mathrm{i}\lambda}h) > 0$ in $\mathbb{D}$. Note that $\mathscr{P}_0$ is known as the Carathéodory class.

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Let $\alpha > 0, \beta \in \mathbb{R}$ and $-\pi/2 < \lambda < \pi/2$. In view of (1.1), for $f \in \mathscr{A}$, we readily see that $f \in \mathscr{B}_\lambda(\alpha, \beta)$ if and only if

$$\operatorname{Re}\left[e^{i\lambda}\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}\right] > 0 \tag{1.2}$$

for some $g \in \mathscr{S}^*$. In this way, the definition of $\mathscr{B}(\alpha, \beta)$ can be extended to the case when $\alpha \geq 0$ naturally. By the above description, we have $\mathscr{B}_0(0,0) = \mathscr{S}^*, \mathscr{B}_\lambda(0,0) = \mathscr{S}_{\mathrm{p}}(\lambda)$ and $\mathscr{B}(1,0) = \mathscr{C}$.

Bazilevič [1] showed that $\mathscr{B}(\alpha, \beta) \subset \mathscr{S}$ for $\alpha > 0, \beta \in \mathbb{R}$. Later, Sheil-Small [5] extended it to the case $\alpha \geq 0$ and gave a geometric characterization for $\mathscr{B}(\alpha, \beta)$. So far, Bazilevič functions form the largest known class in $\mathscr{S}$ which has concrete expressions. It is, however, not easy to study them because the expression is somewhat complicated.

In this paper, we give a few sufficient conditions for a function in $\mathscr{A}$ to belong to a class of Bazilevič functions. Let $-\pi/2 < \lambda < \pi/2$ and set $\zeta = e^{i\lambda}$. We now define the slit domain U_λ by

$$U_\lambda = \mathbb{C} \setminus \{iy : y \geq A_\lambda \text{ or } y \leq -1/A_\lambda\}, \quad A_\lambda = \frac{\cos\lambda}{1 + \sin\lambda}.$$

Note that U_λ is starlike with respect to the origin. In order to state our result, we also introduce the notation

$$P[\gamma, f](z) = 1 + \frac{zf''(z)}{f'(z)} + (\gamma - 1)\frac{zf'(z)}{f(z)}$$

for $\gamma \in \mathbb{C}$ and $f \in \mathscr{A}$.

Theorem 1. *Let $\alpha > 0, \beta \in \mathbb{R}, -\pi/2 < \lambda < \pi/2$ and $f \in \mathscr{A}$. Suppose that*

$$P[\alpha + i\beta, f](z) - \alpha p(z) - i\beta \in U_\lambda, \quad z \in \mathbb{D}, \tag{1.3}$$

holds true for some $p \in \mathscr{P}_0$. Then $f \in \mathscr{B}_\lambda(\alpha, \beta)$.

We have, in particular, the following consequence.

Corollary. *Let $\alpha > 0, \beta \in \mathbb{R}$ and $f \in \mathscr{A}$. Suppose that*

$$\operatorname{Re} P[\alpha + i\beta, f](z) > 0, \quad z \in \mathbb{D}.$$

Then $f \in \mathscr{B}_\lambda(\alpha, \beta)$ for every $\lambda \in (-\pi/2, \pi/2)$.

Indeed, under the assumption of the corollary, $p := (P[\alpha + i\beta, f] - i\beta)/\alpha$ belongs to $\mathscr{P}_0$. Thus, the assumption of Theorem 1 is satisfied for all λ and the claim follows.

Remark 1. If $\beta = 0$, the condition $\operatorname{Re} P[\alpha, f] > 0$ means that f is $(1/\alpha)$-convex (cf. [3, p. 10]). We note that Sakaguchi [4] obtained the stronger result that the condition $\operatorname{Re} P[\gamma, f] > -1/2$ in $\mathbb{D}$ for some $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > -1/2$ is enough to ensure the univalence of f.

Our second result gives a way of constructing many Bazilevič functions from several known functions.

Theorem 2. *Let m be a positive integer, $\alpha_1, \ldots, \alpha_m \in (0, +\infty), \beta \in \mathbb{R}$ and set $\gamma = \alpha_1 + \cdots + \alpha_m + \mathrm{i}\beta$. If $h \in \mathscr{P}_\lambda$ for some $\lambda \in \mathbb{R}$ and $g_1, \ldots, g_m \in \mathscr{S}^*$, then the function f defined by*

$$f(z) = \left[\gamma \int_0^z g_1(\zeta)^{\alpha_1} \ldots g_m(\zeta)^{\alpha_m} h(\zeta)\zeta^{\mathrm{i}\beta-1}\mathrm{d}\zeta\right]^{1/\gamma}, \quad z \in \mathbb{D},$$

belongs to $\mathscr{B}_\lambda(\alpha_1 + \cdots + \alpha_m, \beta)$.

2. Preliminaries

For analytic functions g and h in $\mathbb{D}$, g is said to be *subordinate* to h if there exists an analytic function ω in $\mathbb{D}$ such that

$$\omega(0) = 0, \quad |\omega(z)| < 1 \quad \text{and} \quad g(z) = h(\omega(z)) \quad (z \in \mathbb{D}).$$

This subordination will be denoted by $g \prec h$ or, conventionally, $g(z) \prec h(z)$. In particular, when h is univalent in $\mathbb{D}$, $g \prec h$ if and only if

$$g(0) = h(0) \quad \text{and} \quad g(\mathbb{D}) \subset h(\mathbb{D}).$$

The following result is a key ingredient of the proof of Theorem 1.

Lemma 3 ([3, Cor. 3.1d.1])**.** *Let φ be a non-vanishing analytic function in $\mathbb{D}$ such that $\varphi(0) = 1$ and $z\varphi'(z)/\varphi(z)$ is starlike. Suppose that a non-vanishing analytic function h in $\mathbb{D}$ with $h(0) = 1$ satisfies*

$$\frac{zh'(z)}{h(z)} \prec \frac{z\varphi'(z)}{\varphi(z)}.$$

Then $h \prec \varphi$.

It is convenient to translate the condition (1.2) into one in terms of the quantity $P[\gamma, f]$ for the present aim.

Lemma 4. *Let $\alpha > 0, \beta \in \mathbb{R}, \lambda \in (-\pi/2, \pi/2), f \in \mathscr{A}$ and set $\gamma = \alpha + \mathrm{i}\beta$. Then $f \in \mathscr{B}_\lambda(\alpha, \beta)$ if and only if*

$$P[\gamma, f] = \alpha p + \mathrm{i}\beta + \frac{zh'}{h}$$

for some $p \in \mathscr{P}_0$ and $h \in \mathscr{P}_\lambda$.

Proof. First assume that f is given by (1.1). Then

$$zf(z)^{\gamma-1}f'(z) = g(z)^\alpha h(z) z^{\mathrm{i}\beta}.$$

Taking logarithmic derivatives of both sides and multiplying with z, we obtain the relation

$$1 + \frac{zf''(z)}{f'(z)} + (\gamma - 1)\frac{zf'(z)}{f(z)} = \alpha\frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)} + \mathrm{i}\beta.$$

Since $p = zg'/g \in \mathscr{P}_0$, we have the required condition. We can easily trace back the above procedure by taking $g \in \mathscr{S}^*$ so that $p = zg'/g$. □

3. Proof of theorems

First, we prove the following lemma for the Möbius transformation

$$\psi_\lambda(z) = \frac{1+\bar{\zeta}z}{1-\zeta z} = \frac{1+\mathrm{e}^{-\mathrm{i}\lambda}z}{1-\mathrm{e}^{\mathrm{i}\lambda}z}.$$

Lemma 5. *Let $\lambda \in (-\pi/2, \pi/2)$. Then the above ψ_λ maps the unit disk $\mathbb{D}$ conformally onto the half-plane $\mathrm{Re}\,(\mathrm{e}^{\mathrm{i}\lambda}w) > 0$. Moreover, the function $Q_\lambda(z) = z\psi'_\lambda(z)/\psi_\lambda(z)$ maps $\mathbb{D}$ conformally onto the domain U_λ.*

Proof. For brevity, we set $\psi = \psi_\lambda$ and $Q = Q_\lambda$ for a while. Since $\psi(-\zeta) = 0, \psi(\bar{\zeta}) = \infty$ and $\psi(\mathrm{i}) = \mathrm{i}\bar{\zeta}$, the image of $\partial\mathbb{D}$ under ψ is the line $\mathrm{Re}\,(\zeta w) = 0$. Thus, we see that ψ satisfies $\psi(0) = 1$ and maps the unit disk conformally onto the half-plane $\mathrm{Re}\,(\mathrm{e}^{\mathrm{i}\lambda}w) > 0$.

We next consider the function Q. Since

$$Q(z) = \frac{2z\cos\lambda}{(1+\bar{\zeta}z)(1-\zeta z)},$$

we have the expression

$$\frac{zQ'(z)}{Q(z)} = 1 + \frac{\zeta z}{1-\zeta z} + \frac{-\bar{\zeta}z}{1+\bar{\zeta}z}.$$

Since $\mathrm{Re}\,[z/(1-z)] > -1/2$ for $|z| < 1$, we see that $\mathrm{Re}\,(zQ'(z)/Q(z)) > 0$, equivalently, Q is a starlike univalent function in $\mathbb{D}$.

Finally, for $z = \mathrm{e}^{\mathrm{i}\theta}$, we have

$$Q(z) = \frac{2\cos\lambda}{\bar{z} - z + \bar{\zeta} - \zeta} = \frac{\mathrm{i}\cos\lambda}{\sin\theta + \sin\lambda}.$$

Since

$$-1 + \sin\lambda \le \sin\theta + \sin\lambda \le 1 + \sin\lambda,$$

the boundary values of Q form the set

$$\{\mathrm{i}y : y \ge \cos\lambda/(1+\sin\lambda)\} \cup \{\infty\} \cup \{\mathrm{i}y : y \le -\cos\lambda/(1-\sin\lambda)\}.$$

Therefore, Q maps $\mathbb{D}$ onto the domain U_λ as required. The proof is now complete. □

The function Q_λ is a variant of the so-called "open door mapping" (see [3, §2.5]).

Since $\psi_\lambda(0) = 1$, we observe that $\varphi \in \mathscr{P}_\lambda$ if and only if $\varphi \prec \psi_\lambda$. Hence we obtain the proof of Theorem 1 as follows :

Proof of Theorem 1. Let $\lambda \in (-\pi/2, \pi/2)$ and suppose that $f \in \mathscr{A}$ satisfies (1.3) for some $p \in \mathscr{P}_0$. Take a function $g \in \mathscr{S}^*$ so that $zg'/g = p$. If we put

$$h = \left(\frac{zf'}{f}\right)\left(\frac{f}{g}\right)^{\alpha}\left(\frac{f}{z}\right)^{\mathrm{i}\beta},$$

then the same computation as in the proof of Lemma 4 gives the relation

$$\frac{zh'}{h} = P[\alpha + \mathrm{i}\beta, f] - \alpha p - \mathrm{i}\beta.$$

Thus, by the assumption and Lemma 5, we have $h(z) \neq 0$ for $z \in \mathbb{D}$ and

$$\frac{zh'}{h} \prec Q_\lambda = \frac{z\psi_\lambda'}{\psi_\lambda}.$$

Now it follows from Lemma 3 that $h \prec \psi_\lambda$, i.e., $h \in \mathscr{P}_\lambda$. Hence Lemma 4 implies that $f \in \mathscr{B}_\lambda(\alpha, \beta)$. □

In order to prove Theorem 2, we need the following simple observation.

Lemma 6. *Let $\mu_1, \dots, \mu_m \in (0,1)$ with $\mu_1 + \cdots + \mu_m = 1$ and $g_1, \dots, g_m \in \mathscr{S}^*$. Then the function $g \in \mathscr{A}$ defined by*

$$g(z) = z\left(\frac{g_1(z)}{z}\right)^{\mu_1} \cdots \left(\frac{g_m(z)}{z}\right)^{\mu_m}, \quad z \in \mathbb{D},$$

belongs to $\mathscr{S}^*$.

Proof. By taking the logarithmic derivative of g and multiplying with z, we have

$$\frac{zg'(z)}{g(z)} = \mu_1\frac{zg_1'(z)}{g_1(z)} + \cdots + \mu_m\frac{zg_m'(z)}{g_m(z)}.$$

Since $\mathrm{Re}\,\{zg_j'(z)/g_j(z)\} > 0$ for each j, we conclude that $\mathrm{Re}\,\{zg'(z)/g(z)\} > 0$. □

Remark 2. For a convex subdomain V of $\mathbb{C} \setminus \{0\}$ with $1 \in V$, we let $\mathscr{S}^*(V) = \{f \in \mathscr{A} : zf'(z)/f(z) \in V\ (z \in \mathbb{D})\}$. (For instance, if we choose the half-planes $\mathrm{Re}\, w > \alpha (\geq 0)$ and $\mathrm{Re}\,(e^{\mathrm{i}\lambda}w) > 0$ and the sector $|\arg w| < \pi\alpha/2$ as V, then we have the classes of functions which are starlike of order α, λ-spirallike, and strongly starlike of order α, respectively.) Then, the above proof tells us that $g \in \mathscr{S}^*(V)$ whenever $g_1, \dots, g_m \in \mathscr{S}^*(V)$.

Remark 3. It is well known that, for $g, k \in \mathscr{A}$ with $g(z) = zk'(z)$, the condition $g \in \mathscr{S}^*$ is equivalent to the condition $k \in \mathscr{K}$. Through this transformation, we understand that the above lemma turns to the known fact that the class $\mathscr{K}$ is convex in the sense of Hornich operations (see Cima and Pfaltzgraff [2, Theorem 6.1]).

Finally, as an immediate consequence of the above lemma, we can prove Theorem 2.

Proof of Theorem 2. By letting $\mu_j = \alpha_j/(\alpha_1 + \cdots + \alpha_m)$ and g be as in Lemma 6, we see that $g \in \mathscr{S}^*$ by Lemma 6. Now if we put $\alpha = \alpha_1 + \cdots + \alpha_m$ in the expression (1.1), we have the following expression

$$f(z) = \left[\gamma \int_0^z g_1(\zeta)^{\alpha_1} \cdots g_m(\zeta)^{\alpha_m} h(\zeta)\zeta^{\mathrm{i}\beta - 1}\mathrm{d}\zeta\right]^{1/\gamma} = \left[\gamma \int_0^z g(\zeta)^{\alpha} h(\zeta)\zeta^{\mathrm{i}\beta - 1}\mathrm{d}\zeta\right]^{1/\gamma}.$$

Hence, under the assumptions of Theorem 2, we see that f belongs to $\mathscr{B}_\lambda(\alpha, \beta) = \mathscr{B}_\lambda(\alpha_1 + \cdots + \alpha_m, \beta)$. □

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