A conformal invariant for non-vanishing analytic functions

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Hyperbolic metric.

A plane domain $D \subset \mathbb{C}$ is called **hyperbolic** if $\exists$ an analytic universal covering projection $p : \mathbb{D} \to D$. Here $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The hyperbolic metric $\rho_D(w)|dw|$ of D is determined by the equation

$$\frac{1}{|p'(z)|(1 - |z|^2)} = \rho_D(w), \quad w = p(z).$$

The hyperbolic metric is conformally invariant in the sense that the pull-back

$$f^* \rho_{D'}(z) := \rho_{D'}(f(z))|f'(z)|$$

of $\rho_{D'}(w)|dw|$ under a conformal map $f : D \to D'$ is equal to $\rho_D(z)$.

The quantity $V_D(\varphi)$.

Let φ be a non-vanishing analytic function on a hyperbolic domain D, namely, $\varphi : D \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is holomorphic. Then we set

$$V_D(\varphi) = \sup_{z \in D} \rho_D(z)^{-1} \left| \frac{\varphi'(z)}{\varphi(z)} \right|.$$

Note that $V_D(\varphi)$ can be thought of the Bloch semi-norm of the (possibly multi-valued) function $\log \varphi$.

Subordination.

A holomorphic function f on $\mathbb{D}$ is said to be **subordinate** to another holomorphic function g if there exists a holormorphic function $\omega : \mathbb{D} \to \mathbb{D}$ such that $f = g \circ \omega$ and $\omega(0) = 0$. When we do not require the condition $\omega(0) = 0$ the function f is said to be **weakly subordinate** to g.

When using these notions, the Schwarz-Pick lemma:

$$\frac{|\omega'(z)|}{1 - |\omega(z)|^2} \leq \frac{1}{1 - |z|^2}, \quad \omega : \mathbb{D} \to \mathbb{D},$$

plays an important role. For instance, this leads to the inequality

$$f^*\rho_{D'}(z) \leq \rho_D(z)$$

for a holomorphic map $f : D \to D'$, where equality holds at some (and thus every) point z iff $f : D \to D'$ is a(n unbranched) covering.

Basic properties of $V_D(\varphi)$.

(1) $V_D(\varphi \cdot \psi) \le V_D(\varphi) + V_D(\psi)$.

(2) $V_D(\varphi^\alpha) = |\alpha| V_D(\varphi)$ for $\alpha \in \mathbb{C}$ as long as the power φ^α is defined as a single-valued analytic function on D.

(3) $V_{D_0}(\varphi \circ p) = V_D(\varphi)$ for analytic (unbranched) covering $p : D_0 \to D$.

(4) $V_D(L \circ \varphi) = V_D(\varphi)$ holds for any conformal automorphism L of $\mathbb{C}^*$. In particular, $V_D(1/\varphi) = V_D(\varphi) = V_D(c\varphi)$ for $\forall c \in \mathbb{C}^*$.

(5) $V_{\mathbb{D}}(\varphi) \le V_{\mathbb{D}}(\psi)$ if φ is weakly subordinate to ψ.

(6) $V_D(\varphi) \le V_D(\psi)$ if $\psi : D \to \mathbb{C}^*$ is univalent and if $\varphi(D) \subset \psi(D)$.

Circular width.

Let Ω be a proper subdomain of $\mathbb{C}^*$. Then Ω admits an analytic universal covering projection p of a simply connected proper subdomain D of $\mathbb{C}$ onto it. Then the quantity $W(\Omega) = V_D(p)$ is independent of the particular choice of $p : D \to \Omega$ and will be called the **circular width** of Ω (around the origin). Usually, the domain D is chosen to be the unit disk $\mathbb{D}$ but sometimes another domain is more appropriate to compute the value of $W(\Omega)$.

When Ω is simply connected, we can choose the identity map as p, and thus obtain the relation

$$\frac{1}{W(\Omega)} = \inf_{w\in\Omega} |w|\rho_\Omega(w).$$

Basic properties of $W(\Omega)$.

(1) $W(L(\Omega)) = W(\Omega)$ for any conformal automorphism L of $\mathbb{C}^*$.

(2) $W(\Omega) \leq 4$ for a simply connected domain $\Omega \subset \mathbb{C}^*$.

(3) $W(\Omega) \leq W(\Omega_1)$ if $\Omega \subset \Omega_1 \subset \mathbb{C}^*$.

(4) Let Ω^* denote the circular symmetrization of a domain $\Omega \subset \mathbb{C}^*$. Then $W(\Omega) \leq W(\Omega^*)$.

Applications.

Theorem 1 (Y. C. Kim -S.) *Let* Ω *be a proper subdomain of* $\mathbb{C}^*$. *Suppose that a holomorphic function* f *on* $\mathbb{D}$ *satisfies* $f'(z) \in \Omega$ *for any* $z \in \mathbb{D}$. *If* $W(\Omega) \le 1$, *then* f *is univalent. Moreover, if* $W(\Omega) \le k$, *for some* $0 \le k < 1$, *then* f *extends to a* k*-quasiconformal map of the Riemann sphere.*

Theorem 2 (S. Ponnusamy -S.) *Let* Ω *be a proper subdomain of* $\mathbb{C}^*$. *Suppose that a meromorphic function*

$$F(z) = z + b_0 + b_1/z + b_2/z^2 + \cdots$$

on $|z| > 1$ *satisfies* $F'(z) \in \Omega$. *If* $W(\Omega) \le 1/2$, *then* F *is univalent. Moreover, if* $W(\Omega) \le k/2$, *for some* $0 \le k < 1$, *then* F *extends to a* k*-quasiconformal map of the Riemann sphere.*

Some computations.

(1) **Sector.** $\Omega = \{w \in \mathbb{C} : |\mathrm{arg} w| < \pi\alpha/2\}$

$$\Rightarrow \quad W(\Omega) = 2\alpha.$$

(2) **Annulus.** $\Omega = \{w : r < |w| < R\},\ 0 \le r < R \le \infty$

$$\Rightarrow \quad W(\Omega) = \frac{2}{\pi} \log \frac{R}{r}.$$

(3) **Parallel strip.** $\Omega = \{w : a < \mathrm{Re}\, w < b\},\ 0 < a < b < \infty$

$$\Rightarrow \quad W(\Omega) = \max_{0 \le \theta \le \pi/2} \frac{k \cos\theta}{1 - k\theta/2},$$

where k is a positive number such that

$$\frac{b}{a} = \frac{4 + k\pi}{4 - k\pi}.$$

(4) **Disk.** $\Omega = \{w : |w - a| < r\},\ 0 < r \le a$

$$\Rightarrow \quad W(\Omega) = \frac{2r/a}{1 + \sqrt{1 - (r/a)^2}}.$$

(5) **Wedge.**

$$\Omega = \{w : r < |w| < R, |\text{arg}w| < \pi\alpha/2\}$$

$$\Rightarrow \quad W(\Omega) = \frac{\log(R/r)}{(1+t)\mathcal{K}(t)},$$

where

$$\mathcal{K}(t) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-t^2x^2)}}$$

is the complete elliptic integral of the first kind and $0 < t < 1$ is a number such that

$$\frac{\mathcal{K}(\sqrt{1-t^2})}{\mathcal{K}(t)} = \frac{2\pi\alpha}{\log(R/r)}.$$