COEFFICIENT BOUNDS AND CONVOLUTION PROPERTIES FOR CERTAIN CLASSES OF CLOSE-TO-CONVEX FUNCTIONS

YONG CHAN KIM*), JAE HO CHOI*) AND TOSHIYUKI SUGAWA**)

ABSTRACT. In this paper, we consider the class $\mathcal{C}(\varphi,\psi)$ of normalized close-to-convex functions which is defined by using subordination for analytic functions φ and ψ on the unit disk. Our main object is to provide bounds of the quantity $a_3-\mu a_2^2$ for functions $f(z)=z+a_2z^2+a_3z^3+\cdots$ in $\mathcal{C}(\varphi,\psi)$ in terms of φ and ψ, where μ is a real constant. We also show that the class $\mathcal{C}(\varphi,\psi)$ is closed under the convolution operation by convex functions, or starlike functions of order 1/2 when φ and ψ satisfy some mild conditions.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,$$

which are analytic in the open unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$. Also let $\mathcal{S}$, $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote the subclasses of $\mathcal{A}$ consisting of functions which are univalent, starlike of order α and convex of order α in $\mathbb{D}$. In particular, the classes $\mathcal{S}^*(0)=\mathcal{S}^*$ and $\mathcal{K}(0)=\mathcal{K}$ are the familiar ones of starlike and convex functions in $\mathbb{D}$, respectively. For analytic functions g and h with $g(0)=h(0)$, g is said to be subordinate to h if there exists an analytic function ω on $\mathbb{D}$ such that $\omega(0)=0$, $|\omega(z)|<1$ and $g(z)=h(\omega(z))$ for $z\in\mathbb{D}$. The subordination will be denoted by

$$g\prec h \qquad \text{or} \qquad g(z)\prec h(z) \quad \text{in } \mathbb{D}.$$

Note that $g\prec h$ if and only if $g(0)=h(0)$ and $g(\mathbb{D})\subset h(\mathbb{D})$ when h is univalent in $\mathbb{D}$.

Let $\mathcal{M}$ be the class of analytic functions φ in $\mathbb{D}$ normalized by $\varphi(0)=1$, and let $\mathcal{N}$ be the subclass of $\mathcal{M}$ consisting of those functions φ which are univalent in $\mathbb{D}$ and for which $\varphi(\mathbb{D})$ is convex. Also, for a constant $\alpha\ge 0$, set $\mathcal{N}(\alpha)=\{\varphi\in\mathcal{N}:\operatorname{Re}\varphi>\alpha\}$.

Ma and Minda [6] and the authors [3] defined the subclasses $\mathcal{K}(\varphi)$, $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi,\psi)$ of $\mathcal{A}$ by

$$\mathcal{K}(\varphi)=\left\{f\in\mathcal{A}:1+\frac{zf''(z)}{f'(z)}\prec\varphi(z)\text{ in }\mathbb{D}\right\},$$

$$\mathcal{S}^*(\varphi)=\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec\varphi(z)\text{ in }\mathbb{D}\right\}, \tag{1.1}$$

and

$$\mathcal{C}(\varphi,\psi)=\left\{f\in\mathcal{A}:\exists h\in\mathcal{K}(\varphi)\text{ s.t. }\frac{f'(z)}{h'(z)}\prec\psi(z)\text{ in }\mathbb{D}\right\}$$

for $\varphi,\ \psi\in\mathcal{M}$. Note that $f\in\mathcal{K}(\varphi)$ if and only if $zf'\in\mathcal{S}^*(\varphi)$. Hence $f\in\mathcal{C}(\varphi,\psi)$ if and only if

(1.2)
$$\exists g\in\mathcal{S}^*(\varphi)\text{ such that } zf'(z)/g(z)\prec\psi(z)\text{ in }\mathbb{D}.$$

For functions $\varphi,\ \psi\in\mathcal{M}$, if φ and $e^{-i\beta}\psi$ have positive real part in $\mathbb{D}$, where β is some constant in $(-\pi/2,\pi/2)$, then the class $\mathcal{C}(\varphi,\psi)$ is obviously a subclass of close-to-convex functions, in particular, consists of univalent functions in $\mathbb{D}$. Now we recall that if $f\in\mathcal{A}$ satisfies

$$\left|\arg\frac{zf'(z)}{f(z)}\right|<\frac{\pi}{2}\alpha \qquad (z\in\mathbb{D}) \tag{1.3}$$

for a constant α $(0<\alpha\le 1)$, then $f(z)$ is said to be *strongly starlike of order* α in $\mathbb{D}$, and we write $f\in\mathcal{S}^*_\alpha$. If we set $\varphi_\alpha(z)=((1+z)/(1-z))^\alpha$ $(0<\alpha\le 1)$, then, from (1.1) and (1.3), we can easily see the inclusion

$$\mathcal{S}^*_\alpha=\mathcal{S}^*(\varphi_\alpha)\subset\mathcal{C}(\varphi_\alpha,\varphi_\alpha). \tag{1.4}$$

For constants $\beta\in(-\pi/2,\pi/2)$ and γ with $0\le\gamma<\cos\beta$, we set

$$\psi_{\beta,\gamma}(z)=\frac{1+(e^{i\beta}-2\gamma)e^{i\beta}z}{1-z}.$$

The function $\psi_{\beta,\gamma}$ maps the unit disk onto the half-plane $\{z:\operatorname{Re}(e^{-i\beta}z)>\gamma\}$. Note that $\mathcal{S}^*(\alpha)\equiv\mathcal{S}^*(\psi_{0,\alpha})$ and $\mathcal{K}(\alpha)\equiv\mathcal{K}(\psi_{0,\alpha})$ for $0\le\alpha<1$. Note

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*) Department of Mathematics, College of Education, Yeungnam University, 214-1 Daedong, Gyongsan 712-749, Korea.

**) Department of Mathematics, Kyoto University, 606-8502 Kyoto, Japan.

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also that a function in $\mathcal{S}^*(\psi_{\beta,0})$ is usually called β-spirallike. We set

$$\mathcal{C}_{\alpha,\gamma} = \bigcup_{|\beta|<\arccos\gamma} \mathcal{C}(\psi_{0,\alpha}, \psi_{\beta,\gamma}) \tag{1.5}$$

for $0 \le \alpha < 1$ and $0 \le \gamma < 1$. A function in $\mathcal{C}_{\alpha,\gamma}$ is called *close-to-convex of order* (γ, α) (cf. [2, II, p.89]). In particular, $\mathcal{C} \equiv \mathcal{C}_{0,0}$ is the class of usual close-to-convex functions.

In [3], the second and third authors investigated the norm estimate of the pre-Schwarzian derivatives for the class $\mathcal{C}(\varphi, \psi)$. In this paper, we shall investigate the coefficient bounds of the class $\mathcal{C}(\varphi, \psi)$ and also give convolution properties of functions in $\mathcal{C}(\varphi, \psi)$. Here, the convolution or the Hadamard product $f * g$ of two analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

on $\mathbb{D}$ is defined by

$$(f * g)(z) = f(z) * g(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

2. Preliminary results

The following lemmas will be required in our investigation.

Lemma 2.1. *Assume that $\eta(z) = e_1 + e_2 z + \cdots$ is analytic in $\mathbb{D}$ with $|\eta(z)| \le 1$. Then $|e_1|^2 + |e_2| \le 1$.*

Proof. By Schwarz-Pick's Lemma, we obtain

$$\frac{|\eta'(z)|}{1 - |\eta(z)|^2} \le \frac{1}{1 - |z|^2},$$

so that $|\eta(0)|^2 + |\eta'(0)| \le 1$. Hence $|e_1|^2 + |e_2| \le 1$. □

Lemma 2.2 (Ma and Minda [6]). *Let $\varphi(z) = 1 + A_1 z + A_2 z^2 + \cdots$ be univalent in $\mathbb{D}$ and let $\varphi(\mathbb{D})$ be symmetric with respect to the real axis with $\varphi'(0) > 0$. If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{K}(\varphi)$, then $\left|a_3 - \mu a_2^2\right| \le K(\mu, A_1, A_2)$, where*

$$K(\mu, A_1, A_2) = \begin{cases} (A_2 - \frac{3\mu}{2} A_1^2 + A_1^2)/6 \\ \quad \textit{if } 3A_1^2\mu \le 2(A_2 + A_1^2 - A_1), \\ A_1/6 \\ \quad \textit{if } 2(A_2 + A_1^2 - A_1) \le 3A_1^2\mu \le 2(A_2 + A_1^2 + A_1), \\ (\frac{3\mu}{2} A_1^2 - A_1^2 - A_2)/6 \\ \quad \textit{if } 2(A_2 + A_1^2 + A_1) \le 3A_1^2\mu. \end{cases} \tag{2.1}$$

Lemma 2.3 (Ruscheweyh and Sheil-Small [8]). *Suppose either $g \in \mathcal{K}$, $h \in \mathcal{S}^*$ or else g, $h \in \mathcal{S}^*(1/2)$. Then for any analytic function G in $\mathbb{D}$, we have*

$$\frac{(g * hG)(z)}{(g * h)(z)} \in \overline{\text{co}} G(\mathbb{D}) \qquad (z \in \mathbb{D}),$$

where $\overline{\text{co}} G(\mathbb{D})$ is the closed convex hull of $G(\mathbb{D})$.

3. Main results

We begin by proving

Theorem 3.1. *Let $\varphi(z) = 1 + A_1 z + A_2 z^2 + \cdots$ be univalent in $\mathbb{D}$, $\varphi(\mathbb{D})$ be symmetric with respect to the real axis with $\varphi'(0) > 0$, and let $\psi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ be analytic in $\mathbb{D}$. If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{C}(\varphi, \psi)$, then*

$$\left|a_3 - \mu a_2^2\right| \le K(\mu, A_1, A_2) + M(\mu, A_1, B_1, B_2),$$

where $K(\mu, A_1, A_2)$ is given by (2.1) and

$$M(\mu, A_1, B_1, B_2) = \begin{cases} \frac{1}{3}\left(|B_2 - \frac{3\mu}{4} B_1^2| + A_1 |B_1| |1 - \frac{3\mu}{2}|\right) \\ \qquad \textit{if } A_1 |B_1| |1 - \frac{3\mu}{2}| \ge 2(|B_1| - |B_2 - \frac{3\mu}{4} B_1^2|), \\ \dfrac{|B_1|}{3} + \dfrac{(A_1 |B_1| |1 - \frac{3\mu}{2}|)^2}{12(|B_1| - |B_2 - \frac{3\mu}{4} B_1^2|)} \qquad \textit{otherwise.} \end{cases}$$

Proof. If $f \in \mathcal{C}(\varphi, \psi)$, from the definition of the class $\mathcal{C}(\varphi, \psi)$ there exists a function $h \in \mathcal{K}(\varphi)$ such that $f'/h' \prec \psi$. We set

$$h(z) = z + d_2 z^2 + d_3 z^3 + \cdots$$

and

$$g(z) = \frac{f'(z)}{h'(z)} = 1 + b_1 z + b_2 z^2 + \cdots = \psi(\omega(z)), \tag{3.1}$$

where ω is an analytic function on $\mathbb{D}$ such that $|\omega(z)| \le |z|$ for $z \in \mathbb{D}$. Then a simple calculation shows $b_1 = 2(a_2 - d_2)$ and $b_2 = 3(a_3 - d_3) - 4d_2(a_2 - d_2)$, so that $a_2 = b_1/2 + d_2$ and $a_3 = d_3 + b_2/3 + (2/3) b_1 d_2$. Thus we have

$$\begin{aligned} a_3 - \mu a_2^2 \\ = \left(d_3 - \mu d_2^2\right) + \frac{1}{3}\left(b_2 - \frac{3\mu}{4} b_1^2\right) + \left(\frac{2}{3} - \mu\right) b_1 d_2. \end{aligned} \tag{3.2}$$

By Lemma 2.2, we have

$$|d_3 - \mu d_2^2| \le K(\mu, A_1, A_2). \tag{3.3}$$

We write $\omega(z) = e_1 z + e_2 z^2 + \cdots$. Then, from (3.1) we have $b_1 = B_1 e_1$ and $b_2 = B_1 e_2 + B_2 e_1^2$. Since

$1+\frac{zh''(z)}{h'(z)} \prec \varphi(z)$ in $\mathbb{D}$, Rogosinski's result [7] implies $|d_2| \le \frac{1}{2}A_1$. Therefore, we get

$$\begin{aligned}&\left|\frac{1}{3}\left(b_2-\frac{3\mu}{4}b_1^2\right)+\left(\frac{2}{3}-\mu\right)b_1d_2\right|\\ &\le \frac{|B_1|}{3}|e_2|+\frac{1}{3}\left|B_2-\frac{3\mu}{4}B_1^2\right||e_1|^2\\ &\quad+\left|\frac{2}{3}-\mu\right||d_2B_1||e_1|\\ &\le \frac{|B_1|}{3}|e_2|+\frac{1}{3}\left|B_2-\frac{3\mu}{4}B_1^2\right||e_1|^2\\ &\quad+\left|\frac{1}{3}-\frac{\mu}{2}\right|A_1|B_1||e_1|.\end{aligned}$$

Taking $\eta(z)=\omega(z)/z$ in Lemma 2.1, we obtain $|e_2| \le 1-|e_1|^2$, so that

$$\left|\frac{1}{3}\left(b_2-\frac{3\mu}{4}b_1^2\right)+\left(\frac{2}{3}-\mu\right)b_1d_2\right| \le P(|e_1|),$$

where $P(x)=ax^2+bx+c$ and $a=\frac{1}{3}(|B_2-\frac{3\mu}{4}B_1^2|-|B_1|)$, $b=A_1|B_1||\frac{1}{3}-\frac{\mu}{2}|$ and $c=|B_1|/3$. Since $b \ge 0$ and $0 \le |e_1| \le 1$, we have

$$P(|e_1|) \le \begin{cases} P(-b/2a)=c-b^2/4a \\ \quad \text{if } a<0 \text{ and } -b/2a<1, \\ P(1)=a+b+c \qquad \text{otherwise.}\end{cases}$$

Thus we conclude

$$\begin{aligned}&\left|\frac{1}{3}\left(b_2-\frac{3\mu}{4}b_1^2\right)+\left(\frac{2}{3}-\mu\right)b_1d_2\right| \\ &\le M(\mu,A_1,B_1,B_2).\end{aligned} \tag{3.4}$$

Hence, making use of (3.3) and (3.4) in equality (3.2), we obtain the desired result. □

Corollary 3.2. *If $f(z)=z+a_2z^2+a_3z^3+\cdots \in \mathcal{C}(\psi_{0,0},\ \psi_{0,0})$, then*

$$|a_3-\mu a_2^2| \le \begin{cases} 3-4\mu & \text{if } \mu \le 1/3,\\ 1/3+4/9\mu & \text{if } 1/3 \le \mu \le 2/3,\\ (16-21\mu+9\mu^2)/3(4-3\mu) & \\ \quad \text{if } 2/3 \le \mu \le 1 & \\ 3\mu-5/3 & \text{if } 1 \le \mu \le 4/3,\\ 4\mu-3 & \text{if } 4/3 \le \mu.\end{cases}$$

Remark . ¿From (1.5) it is clear that $\mathcal{C}(\psi_{0,0},\ \psi_{0,0}) \subset \mathcal{C}$. For the cases of $0 \le \mu \le 1/3$ and $1/3 \le \mu \le 2/3$, the above estimates agree with those of Koepf [4].

If we take $\varphi=\psi=\varphi_\alpha=z+2\alpha z^2+2\alpha^2z^3+\cdots$ in Theorem 3.1, we obtain

Corollary 3.3. *If $f(z)=z+a_2z^2+a_3z^3+\cdots \in \mathcal{C}(\varphi_\alpha,\varphi_\alpha)$, then*

$$|a_3-\mu a_2^2| \le \begin{cases} (3-4\mu)\alpha^2 & \text{if } 3\alpha\mu \le 2\alpha-1\\ (1-\mu)\alpha^2+\dfrac{\alpha}{3}\left\{2+\dfrac{(2-3\mu)^2\alpha^2}{2-(2-3\mu)\alpha}\right\} & \\ \quad \text{if } 2\alpha-1 \le 3\alpha\mu \le 3\alpha-1 & \\ \alpha\left\{1+\dfrac{(2-3\mu)^2\alpha^2}{3(2-2\alpha+3\alpha\mu)}\right\} & \\ \quad \text{if } 3\alpha-1 \le 3\alpha\mu \le 2\alpha & \\ \alpha\left\{1+\dfrac{(2-3\mu)^2\alpha^2}{3(2-3\alpha\mu+2\alpha)}\right\} & \\ \quad \text{if } 2\alpha \le 3\alpha\mu \le 2\alpha+1 & \\ (3\mu-2)\alpha^2+\alpha/3 & \\ \quad \text{if } 2\alpha+1 \le 3\alpha\mu \le 3\alpha+1 & \\ (4\mu-3)\alpha^2 & \text{if } 3\alpha+1 \le 3\alpha\mu.\end{cases}$$

Noting the relation $\mathcal{S}^*_\alpha \subset \mathcal{C}(\varphi_\alpha,\varphi_\alpha)$, we would have an estimate for strongly starlike functions of order α. When $3\alpha\mu \le 2\alpha-1$ or $3\alpha\mu \ge 3\alpha+1$, that estimate incidentally coincides with the sharp estimate for strongly starlike functions of order α obtained previously by Ma and Minda [5].

Now, by using Lemma 2.3, we investigate convolution properties of functions in $\mathcal{C}(\varphi,\psi)$. First, we recall results due to Ma and Minda. The following form is slightly different from the original one, so we include its proof here.

Proposition 3.4 ([6]).

(a) *Let $\varphi \in \mathcal{N}(0)$. For $g \in \mathcal{K}$ and $h \in \mathcal{S}^*(\varphi)$, we have $g*h \in \mathcal{S}^*(\varphi)$.*

(b) *Let $\varphi \in \mathcal{N}(1/2)$. For $g \in \mathcal{S}^*(1/2)$ and $h \in \mathcal{S}^*(\varphi)$, we have $g*h \in \mathcal{S}^*(\varphi)$.*

Proof. First, we prove (a). Set $G=zh'/h \prec \varphi$. Since $z(g*h)'=g*(zh')=g*(Gh)$, from Lemma 2.3, we see

$$\frac{z(g*h)'(z)}{(g*h)(z)}=\frac{(g*Gh)(z)}{(g*h)(z)} \in \overline{\mathrm{co}}G(\mathbb{D}) \subset \overline{\varphi(\mathbb{D})}.$$

Hence, we have $z(g*h)'/g*h \prec \varphi$. Assertion (b) can be shown similarly. □

With the aid of the above result, we can now prove the following.

Theorem 3.5.

(a) *Let $\varphi \in \mathcal{N}(0)$ and $\psi \in \mathcal{N}$. Then, for $g \in \mathcal{K}$ and $f \in \mathcal{C}(\varphi,\psi)$, we have $g*f \in \mathcal{C}(\varphi,\psi)$.*

(b) *Let $\varphi \in \mathcal{N}(1/2)$ and $\psi \in \mathcal{N}$. Then, for $g \in \mathcal{S}^*(1/2)$ and $f \in \mathcal{C}(\varphi,\psi)$, we have $g*f \in \mathcal{C}(\varphi,\psi)$.*

Proof. We show only (a). We can handle (b) in the same fashion. Let $\varphi \in \mathcal{N}(0)$ and $\psi \in \mathcal{N}$. If $f \in \mathcal{C}(\varphi, \psi)$, there is a function $h \in \mathcal{S}^*(\varphi)$ such that $zf'/h \prec \psi$. Set $G(z) = zf'(z)/h(z)$. Then $G(\mathbb{D}) \subset \psi(\mathbb{D})$ and $z(g*f)' = g*(zf') = g*Gh$. Since $\psi(\mathbb{D})$ is convex and since $z(g*f)'/(g*h)$ is analytic, Lemma 2.3 implies that

$$\frac{z(g*f)'(z)}{(g*h)(z)} = \frac{(g*Gh)(z)}{(g*h)(z)}$$

lies in $\psi(\mathbb{D})$, in other words, $z(g*f)'/g*h \prec \psi$. Now Proposition 3.4 ensures $g*h \in \mathcal{S}^*(\varphi)$. Hence we find from definition (1.2) that $g*f \in \mathcal{C}(\varphi, \psi)$, which completes the proof of Theorem 3.5. □

Remark . If we apply the above theorem to the case $\varphi = \psi_{0,0}$ and $\psi = \psi_{\beta,0}$ for $|\beta| < \pi/2$, then Theorem 3.5 would immediately yield that $f*g \in \mathcal{C}$ for $f \in \mathcal{C}$ and $g \in \mathcal{K}$ (see [1, Theorem 8.7]).

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