Geometric properties of functions with small Schwarzian derivatives

Toshiyuki Sugawa

Department of Mathematics,
Graduate School of Science,
Hiroshima University,
Higashi-Hiroshima, 739-8526 Japan
`sugawa@math.sci.hiroshima-u.ac.jp`
`http://sugawa@cajpn.org/`

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Ji-A Kim at POSTECH, Korea

Schwarzian derivative.

For a non-constant meromorphic function f in a plane domain, the Schwarzian derivative S_f of f is defined by

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 .$$

S_f is holomorphic at z_0 $\Leftrightarrow$ f is locally univalent at z_0.

Fact: $S_f = 0$ if and only if f is (a restriction of) a Möbius map. Moreover, $S_{L\circ f} = S_f$ for any Möbius transformation L.

$\Longrightarrow$ Schwarzian derivative measures deviation of the function from Möbius maps.

Univalence criteria.

Theorem 1 (Nehari $\sim$ 1949) *If f is univalent meromorphic in the unit disk $\mathbb{D}$, then*

$$|S_f(z)| \le 6(1-|z|^2)^{-2}.$$

Conversely, if a meromorphic function f in $\mathbb{D}$ satisfies

$$|S_f(z)| \le 2(1-|z|^2)^{-2},$$

then f is univalent in $\mathbb{D}$. The numbers 6 and 2 are sharp.

• The Koebe function $K(z) = z/(1-z)^2$ satisfies

$$S_K(z) = \frac{-6}{(1-z^2)^2}.$$

• The function $L(z) = (1/2)\log(1+z)/(1-z)$ which maps $\mathbb{D}$ onto the paralell strip $|\mathrm{Im}\, w| < \pi/2$ satisfies

$$S_L(z) = \frac{2}{(1-z^2)^2}.$$

Theorem 2 (Nehari 1949, Pokornyi 1951) *If f satisfies one of the following conditions in $\mathbb{D}$, then f is univalent in $\mathbb{D}$:*

$$|S_f(z)| \le \frac{\pi^2}{2},$$
$$|S_f(z)| \le 4(1-|z|^2)^{-1}.$$

These numbers are sharp.

- Extremal functions are given, respectively, by

$$\tan\frac{\pi z}{2} \quad \text{and} \quad \frac{z}{2(1-z^2)} + \frac{1}{4}\log\frac{1+z}{1-z}.$$

Based on these results, a more general univalence criteria were deduced by Avkhadiev and *et al*.

Connection with a linear ODE.

For a given holomorphic function φ in the unit disk $\mathbb{D}$, we can construct a locally univalent meromorphic function f so that $S_f = \varphi$ in $\mathbb{D}$. Indeed, let y_0 and y_1 be the analytic solutions to the ODE

$$2y'' + \varphi y = 0$$

in $\mathbb{D}$ with the initial conditions

$$\begin{aligned} y_0(0) &= 1, \quad & y_1(0) &= 0, \\ y_0'(0) &= 0, \quad & y_1'(0) &= 1. \end{aligned}$$

Note: the Wronskian is identically 1 :

$$y_0 y_1' - y_0' y_1 \equiv 1.$$

Then the quotient $f = y_1/y_0$ is a desired one, because the logarithmic derivative of

$$f' = \frac{y_0 y_1' - y_0' y_1}{y_0^2} = \frac{1}{y_0^2}$$

yields

$$\frac{f''}{f'} = -\frac{2y_0'}{y_0}.$$

Hence,

$$\begin{aligned} S_f &= \left(-\frac{2y_0'}{y_0}\right)' - \frac{1}{2}\left(-\frac{2y_0'}{y_0}\right)^2 \\ &= -\left(\frac{2y_0''}{y_0}\right) + 2\left(\frac{y_0'}{y_0}\right)^2 - 2\left(\frac{y_0'}{y_0}\right)^2 \\ &= \varphi. \end{aligned}$$

Note: the constructed function f satisfies $f(0) = 0, f'(0) = 1$ and $f''(0) = 0$. Third condition can easily be missed!

Normalizations.

$\mathcal{M}$: the set of meromorphic functions f in the unit disk $\mathbb{D}$ with $f(0) = 0, f'(0) = 1$. For a complex number c, set

$$\mathcal{M}(c) = \{f \in \mathcal{M} : f''(0) = 2c\}.$$

Fact:*for φ and for $c \in \mathbb{C}$, there is the unique function $f = f_{\varphi,c}$ in $\mathcal{M}(c)$ for which $S_f = \varphi$ holds.*

Indeed, such an f can be given by

$$f_{\varphi,c} = \frac{y_1}{y_0 - cy_1} = \frac{f_{\varphi,0}}{1 - cf_{\varphi,0}}.$$

Note: $f_{\varphi,c}(z) = z + cz^2 + \cdots$.

Omitted values.

Set $K(\varphi) = \{c \in \mathbb{C} : 1/c \notin f_{\varphi,0}(\mathbb{D})\}$. Note that $K(\varphi)$ is always compact.

- $f_{\varphi,c}$ is pole-free (i.e., analytic) $\Leftrightarrow c \in K(\varphi)$.

- $|c| \leq 2$ for each $c \in K(\varphi)$ if $f_{\varphi,0}$ is univalent meromorphic.

Weight functions.

$A(x)$, $0 \leq x < 1$, is called a *weight function* if it is locally Lipschitz, non-decreasing, and positive.

Example: $A(x) = C(1 - x^2)^{-\mu}$.

Let U_0, U_1, V_0 and V_1 be the functions on $[0, 1)$ determined by

$$
\begin{aligned}
2U_0'' &= AU_0, & U_0(0) &= 1, & U_0'(0) &= 0,\\
2U_1'' &= AU_1, & U_1(0) &= 0, & U_1'(0) &= 1,\\
2V_0'' &= -AV_0, & V_0(0) &= 1, & V_0'(0) &= 0,\\
2V_1'' &= -AV_1, & V_1(0) &= 0, & V_1'(0) &= 1.
\end{aligned}
$$

When we need to indicate the weight function A, we write, for example, $U_0(x, A) = U_0(x)$.

• $U_0 > 0$ and $U_0' > 0$ hold on the interval $[0, 1)$ for any weight function A.

A general univalence criterion

Theorem 3 (Nehari 1954) *If*
(i) $A(x)(1-x^2)^2$ *is non-increasing in* $0 \le x < 1$,
(ii) $V_0(x, A)$ *is positive for* $0 \le x < 1$,
then the condition $|S_f(z)| \le A(|z|)$ *for a function* $f \in \mathcal{M}$ *implies univalence of* f *in* $\mathbb{D}$.

Examples:
For $A(x) = \pi^2/2$, one has $V_0(x) = \cos(\pi x/2)$.
For $A(x) = 4(1-x^2)^{-1}$, one has $V_0(x) = 1-x^2$.
For $A(x) = 2(1-x^2)^{-2}$, one has $V_0(x) = \sqrt{1-x^2}$.

Problem:
What geometric properties can we say about those functions with prescribed growth for the Schwarzian derivatives?

Starlike and convex functions.

$f \in \mathcal{M}$ is called *starlike* if f is univalent analytic and the image $f(\mathbb{D})$ is starlike with respect to the origin, in other words,

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} > 0, \quad |z| < 1.$$

$f \in \mathcal{M}$ is called *convex* if f is univalent analytic and the image $f(\mathbb{D})$ is convex, in other words,

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad |z| < 1.$$

For a constant $\alpha \in [0, 1)$, a function $f \in \mathcal{M}$ is called *starlike of order* α if

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} > \alpha, \quad |z| < 1.$$

• Strohhäcker theorem: *a convex function is starlike of order* $1/2$.

• Starlike functions of order $1/2$ play an important role in the theory of convolution (Hadamard product).

• These properties are not preserved by post-composition of Möbius maps unlike univalence.

Starlikeness Theorem.

Theorem 4 *Let A be a weight function and c be a complex number. Suppose*

$$2\int_0^1 U_0{}'(x)U_1(x)dx + |c|U_1(1)^2 \le 1.$$

If a function $f \in \mathcal{M}(c)$ satisfies $|S_f(z)| \le A(|z|)$ in $|z| < 1$, then f is starlike of order $1/2$.

As the special case when A is a positive constant and $c = 0$, we obtain

Corollary 5 *Let* $C_0 = 2\beta_0^2 \approx 2.37036$, *where* β_0 *is the unique positive root of the equation* $\sinh(2\beta) = 4\beta$. *If a function* $f \in \mathcal{M}(0)$ *satisfies the inequality* $|S_f(z)| \le C_0$ *in* $|z| < 1$, *then* f *is a starlike function of order* $1/2$. *The constant* C_0 *is sharp.*

Remarks:

Gabriel (1955) proved that $|S_f(z)| \le C_0'$ implies starlikeness of $f \in \mathcal{M}(0)$, where $C_0' = 2{\beta_0'}^2 \approx 2.71707$ and β_0' is the unique root of the equation $2\beta = \tan\beta$ in $0 < \beta < \pi/2$.

On the other hand, Chiang (1994) showed that C_0' cannot be replaced by a larger number than $C_0'' = (\xi^2 + \eta^2)/2 \approx 4.6351$, where ξ and η are the smallest positive roots of the equations $\xi \tan\xi = -1$ and $\eta \tanh\eta = 1$.

By some experiments, it is likely that C_0'' is the best possible constant.

Convexity Theorem.

Theorem 6 *Let A be a weight function and c be a complex number. Suppose that the functions V_0 and V_1 satisfy the inequalities*

$$V_0(x) - |c|V_1(x) > 0, \ 0 \le x < 1,$$

and

$$V_0(1) + 2{V_0}'(1) \ge |c|(V_1(1) + 2{V_1}'(1)).$$

If a function $f \in \mathcal{M}(c)$ satisfies $|S_f(z)| \le A(|z|)$ in $|z| < 1$, then f is convex.

As a corollary, we obtain an improvement of a result of Chiang (1994).

Corollary 7 *Let* $C_1 = 2\beta_1^2 \approx 0.853526$, *where* β_1 *is the unique root of the equation* $2\beta \tan\beta = 1$ *in* $0 < \beta < \pi/2$. *If a function* $f \in \mathcal{M}(0)$ *satisfies the inequality* $|S_f(z)| \le C_1$ *in* $|z| < 1$, *then* f *is a convex function.*

Remarks:

The constant C_1 above is not sharp. More precisely, C_1 is the sharp constant for which $|S_f(z)| \le C_1$ implies the inequality $|zf''(z)/f'(z)| < 1$ in $|z| < 1$.

As Chiang (1994) showed, the constant C_1 cannot be replaced by a larger number than $C_1' = 2{\beta_1'}^2 \approx 1.19105$, where β_1' is the unique positive root of the equation $\beta \tanh\beta = 1/2$.

Growth theorems.

Our main theorems are based on some growth theorems for solutions to the ODE introduced earlier.

Lemma 8 *Let A be a weight function and suppose that $|\varphi(z)| \leq A(|z|)$ in $|z| < 1$. The solutions y_0 and y_1 to the differential equation $2y'' + \varphi y = 0$ in $\mathbb{D}$ with the initial conditions $y_0(0) = 1, y_0'(0) = 0, y_1(0) = 0, y_1'(0) = 1$ then satisfy the inequalities*

$$\begin{aligned} \tilde{V_0}(|z|, A) \leq & |y_0(z)| \leq U_0(|z|, A), \\ & |y_0'(z)| \leq {U_0}'(|z|, A), \\ \tilde{V_1}(|z|, A) \leq & |y_1(z)| \leq U_1(|z|, A), \\ & |y_1'(z)| \leq {U_1}'(|z|, A) \end{aligned}$$

for $z \in \mathbb{D}$, where $\tilde{V}(x) = V(x)$ for $0 \leq x < x_0$ and $\tilde{V}(x) = 0$ for $x \geq x_0$ and x_0 is the smallest positive zero of $V(x)$ (if there is no such zero, set $x_0 = 1$).

Lemma 9 *Under the same hypothesis as in the previous lemma, let* $y_2 = y_0 - cy_1$, *where* c *is a complex constant for which the function* $V_2 = V_0 - |c|V_1$ *is positive on* $(0,1)$. *Then the inequality*

$$\left|\frac{y_2'(z)}{y_2(z)}\right| \leq -\frac{V_2{}'(|z|)}{V_2(|z|)}$$

holds for every $z \in \mathbb{D}$.

Idea of proof:

For a fixed $\zeta \in \partial\mathbb{D}$, we set $w(t) = y_2'(t\zeta)/y_2(t\zeta)$ and $v(t) = -V_2'(t)/V_2(t)$. Then, the function w satisfies the Riccati equation

$$w' = -\frac{\varphi}{2} - w^2.$$

Hence, the function $u(t) = |w(t)|$ satisfies the differential inequality

$$u' \leq |w'| \leq \frac{A}{2} + u^2.$$

Similarly, the function v satisfies $v' = A/2 + v^2$. Use a comparison theorem!

A comparison theorem.

The following is a specialized comparison theorem for the present situation.

Lemma 10 (cf. Walter, "ODE", p. 96)
Let A be a non-negative continuous function on $[0,1)$ *and set $Pw = w' - A/2 - w^2$. If absolutely continuous real-valued functions u, v on* $[0,1)$ *satisfy the inequalities*
(a) $Pu \le Pv$ a.e. in $[0,1)$ and
(b) $u(0) \le v(0)$,
then $u \le v$ holds in $[0,1)$.

Proof of Starlikeness Theorem. Let $f = y_1/y_2$, where $y_2 = y_0 - cy_1$. Then the quantity $p(z) = zf'(z)/f(z)$ satisfies

$$\frac{1}{p(z)} = \frac{y_1(z)y_2(z)}{z} = \int_0^1 \Big(y_1 y_2\Big)'(tz)dt = 1 + 2\int_0^1 y_1(tz)y_2'(tz)dt.$$

We now use the growth theorem to get

$$\left|\frac{1}{p(z)} - 1\right| \le 2\int_0^1 U_1(t|z|){U_2}'(t|z|)dt \le 2\int_0^1 U_1(t){U_2}'(t)dt = 2\int_0^1 U_1(t){U_0}'(t)dt + |c|U_1(1)^2.$$

We now conclude that $|1/p(z) - 1| < 1$, which is equivalent to $\operatorname{Re} p(z) > 1/2$.

Proof of Convexity Theorem.

Use the same notation as in the previous proof. Further we set $V_2 = V_0 - |c|V_1$. Then, since $f' = y_2^{-2}$,

$$1 + \frac{zf''(z)}{f'(z)} = 1 - 2\,\frac{y_2'(z)}{y_2(z)}.$$

By the second growth lemma,

$$\left|2\,\frac{y_2'(z)}{y_2(z)}\right| \le -2\,\frac{{V_2}'(|z|)}{V_2(|z|)} < -2\,\frac{{V_2}'(1)}{V_2(1)}$$

The last term is certainly not greater than 1. Therefore, $\mathrm{Re}\,(1 + zf''(z)/f'(z)) > 0$.

Computations.

• The simplest case is when A is a positive constant. If we write $A = 2\beta^2$, where β is a positive number, then

$$\begin{aligned} U_0(x) &= \cosh(\beta x), \\ U_1(x) &= \sinh(\beta x)/\beta, \\ V_0(x) &= \cos(\beta x), \\ V_1(x) &= \sin(\beta x)/\beta. \end{aligned}$$

• For $A(x) = C(1 - x^2)^{-2}$, where the constant C is allowed to be negative for convenience. If we write $C = 2(4\alpha^2 - 1)$, then

$$\begin{aligned} U_0(x) &= \sqrt{1 - x^2}\cosh\left[\alpha \log\left(\frac{1+x}{1-x}\right)\right], \\ U_1(x) &= \frac{\sqrt{1 - x^2}}{2\alpha}\sinh\left[\alpha \log\left(\frac{1+x}{1-x}\right)\right]. \end{aligned}$$

• For $A(x) = C(1-x^2)^{-1}$ with positive constant $C = (1-\alpha^2)/2$,

$$U_0(x) = F(-\tfrac{1+\alpha}{4}, -\tfrac{1-\alpha}{4}; \tfrac{1}{2}; x^2)$$
$$U_1(x) = x\, F(\tfrac{1+\alpha}{4}, \tfrac{1-\alpha}{4}; \tfrac{3}{2}; x^2),$$

where $F(a, b; c; x)$ stands for the hypergeometric function.

As a corollary, we get

Corollary 11 *Let $C_2 = (1+\beta_2^2)/2 \approx 1.52444$, where β_2 is the unique positive root of the equation*

$$\int_0^1 x^2 F(\tfrac{3+i\beta}{4}, \tfrac{3-i\beta}{4}; \tfrac{3}{2}; x^2) F(\tfrac{1+i\beta}{4}, \tfrac{1-i\beta}{4}; \tfrac{3}{2}; x^2) dx$$
$$= \frac{2}{1+\beta^2}.$$

If a function $f \in \mathcal{M}(0)$ satisfies the inequality $|S_f(z)| \le C_2/(1-|z|^2)$ in $|z| < 1$, then f is a starlike function of order $1/2$. The constant C_2 is sharp.

Final remark.

The Schwarzian radius of convexity (cf. Chiang 1994) must be zero unless we impose some restriction on the second coefficient $c = f''(0)/2$.

Proposition 12 *Let φ be analytic in the unit disk. Suppose that $f_{\varphi,0}$ is univalent and that $f_{\varphi,c}$ is convex for every $c \in K(\varphi)$. Then, $\varphi = 0$.*

This follows from the more geometric assertion.

Proposition 13 *Let D be a proper subdomain of the complex plane $\mathbb{C}$. Suppose that $L(D)$ is convex for each Möbius transformation L such that $L^{-1}(\infty) \notin D$. Then D is a disk or a half-plane.*