ON CONFORMAL MAPPINGS ONTO THE INTERIOR OF CONIC SECTIONS

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1. Introduction

If a univalent function $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ has positive Taylor coefficients at the origin, various sharp estimates can be easily deduced. For example, one can show the inequalities

$$|f(z) - a_0 - a_1 z - \cdots - a_k z^k| \le f(|z|) - a_0 - a_1|z| - \cdots - a_k|z|^k$$

and

$$|f^{(k)}(z)| \le f^{(k)}(|z|)$$

for $k = 0, 1, 2, \ldots$.

As one immediately sees, a necessary condition for a univalent function f to have positive Taylor coefficients is that the image domain $\Omega = f(\mathbb{D})$ is symmetric in the real axis. However, under the symmetricity assumption, it seems to be difficult to give a sufficient condition for that in terms of the shape of Ω. For instance, the convexity of Ω is not sufficient. In fact, for constants $0 < c < 1$ and $\alpha \in (1, \infty) \setminus \mathbb{Z}$ with $c\alpha \le 1$ the function

$$f(z) = (1 + cz)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} (cz)^n$$

is univalent in $\mathbb{D}$ and has convex image because

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + (\alpha - 1)\mathrm{Re}\,\frac{cz}{1 + cz} > 1 - (\alpha - 1)\frac{c}{1 - c} \ge 0.$$

Note that $\binom{\alpha}{n} > 0$ when $n < \alpha + 1$ and $\binom{\alpha}{n} < 0$ when $\alpha + 1 < n < \alpha + 2$.

In this talk, we will explain an approach to show positivity of the Taylor coefficients of a specific conformal mapping of the interior of a conic section.

2. Conformal mappings onto domains bounded by conic sections

For $k \in [0, \infty)$, we set

$$\Omega_k = \{u + iv \in \mathbb{C}; u^2 > k^2(u - 1)^2 + k^2 v^2, u > 0\}.$$

Note that $1 \in \Omega_k$ for all k. Ω_0 is nothing but the right half plane. When $0 < k < 1$, Ω_k is the unbounded domain enclosed by the right half of the hyperbola

$$\left(\frac{u + k^2/(1 - k^2)}{k/(1 - k^2)}\right)^2 - \frac{v^2}{1/(1 - k^2)} = 1$$

with focus at 1. Ω_1 becomes the unbounded domain enclosed by the parabola

$$v^2 = 2u - 1$$

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The present work grew out of discussions with Stanisława Kanas.

with focus at 1. When $k > 1$, the domain Ω_k is the interior of the ellipse

$$\left(\frac{u + k^2/(1-k^2)}{k/(1-k^2)}\right)^2 + \frac{v^2}{1/(k^2-1)} = 1$$

with focus at 1. For every k, the domain Ω_k is convex and symmetric in the real axis. Note also that $\Omega_{k_1} \supset \Omega_{k_2}$ if $0 \le k_1 \le k_2$.

Kanas and Wisniowska [2] treated the family Ω_k in their study of k-uniformly convex functions and gave the explicit formulae for the conformal homeomorphisms $p_k : \mathbb{D} \to \Omega_k$ determined by $p_k(0) = 1$ and $p_k'(0) > 0$.

In order to state their result, we prepare some notation. Let $\mathcal{K}(z,t)$ and $\mathcal{K}(t)$ be the normal and complete elliptic integral, respectively, i.e.,

$$\mathcal{K}(z,t) = \int_0^z \frac{dx}{\sqrt{(1-x^2)(1-t^2x^2)}}$$

and $\mathcal{K}(t) = \mathcal{K}(1,t)$. The quantity

$$\mu(t) = \frac{\pi\mathcal{K}(\sqrt{1-t^2})}{2\mathcal{K}(t)}$$

is known as the modulus of the Groetszch ring $\mathbb{D} \setminus [0,t]$ for $0 < t < 1$. Note that $\mu(t)$ is a strictly decreasing smooth function. For details, see [1].

Proposition 1 (Kanas-Wisniowska [2]). *The conformal map $p_k : \mathbb{D} \to \Omega_k$ with $p_k(0) = 1$ and $p_k'(0) > 0$ is given by*

$$p_k(z) = \begin{cases} (1+z)/(1-z) & \text{if } k = 0, \\ (1-k^2)^{-1}\cosh[C_k \log(1+\sqrt{z})/(1-\sqrt{z})] - k^2/(1-k^2) & \text{if } 0 < k < 1, \\ 1 + (2/\pi^2)[\log(1+\sqrt{z})/(1-\sqrt{z})]^2 & \text{if } k = 1, \\ (k^2-1)^{-1}\sin[C_k\mathcal{K}((z/\sqrt{t}-1)/(1-\sqrt{t}z), t)] + k^2/(k^2-1) & \text{if } 1 < k, \end{cases}$$

where $C_k = (2/\pi)\arccos k$ for $0 < k < 1$ and $C_k = \pi/2\mathcal{K}(t)$ and $t \in (0,1)$ is chosen so that $k = \cosh(\mu(t)/2)$ for $k > 1$.

3. Main Results

For each $k \in [0, \infty)$, we write

$$p_k(z) = 1 + A_1(k)z + A_2(k)z^2 + \cdots$$

for the conformal mapping p_k of $\mathbb{D}$ onto Ω_k with $p_k(0) = 1$ and $p_k'(0) > 0$. By Carathéodory's theorem, $|A_n(k)| \le 2$ holds for each $n \ge 1$ and $k \in [0, \infty)$. Our main result is the following.

Theorem 2. *$A_n(k) \ge 0$ for all $n \ge 1$ and $k \in [0,1]$.*

Conjecture 1. *$A_n(k) \ge 0$ for all $n \ge 1$ and $k \in [0, \infty)$.*

Since $p_0(z) = 1 + 2z + 2z^2 + 2z^3 + \cdots$ and

$$p_1(z) = 1 + \frac{2}{\pi^2}\left(z + \frac{z^2}{3} + \frac{z^3}{5} + \cdots\right)^2,$$

the assertion of the theorem is trivial for $k = 0$ and $k = 1$. In what follows, we consider the cases when $0 < k < 1$ and $k > 1$. Due to complexity of the representations of p_k given above, we try to simplify them.

We now consider the conformal mapping J of $\mathbb{D}$ onto $\widehat{\mathbb{C}} \setminus [-1,1]$ defined by $f(z) = (z + z^{-1})/2$. Since

$$J(e^{-s+it}) = \cosh s \cos t - i \sinh s \sin t,$$

the circle $|z| = e^{-s}$ is mapped by J onto the ellipse E_s given by

$$\left(\frac{u}{\cosh s}\right)^2 + \left(\frac{v}{\sinh s}\right)^2 = 1$$

for $s > 0$ and the radial segment $(0, e^{it})$ is mapped by J into the component H_t of the hyperbola given by

$$\left(\frac{u}{\cos t}\right)^2 - \left(\frac{v}{\sin t}\right)^2 = 1, \quad u\cos t > 0,$$

for $t \in \mathbb{R}$ with $(2/\pi)t \notin \mathbb{Z}$.

Let T_n be the Chebyshev polynomial of degree n, i.e., $T_n(\cos\theta) = \cos(n\theta)$. Then it is well known that the n-fold mapping $z \mapsto z^n$ is conjugate under J to T_n, in other words,

$$J(z^n) = T_n(J(z))$$

holds in $|z| < 1$. In particular, one can see that the ellipse E_s is mapped by T_n onto E_{ns} and that the hyperbola H_t is mapped by T_n onto H_{nt}.

Applying the above argument to $T_2(w) = 2w^2 - 1$, we obtain the following.

Lemma 3. *The Chebyshev polynomial $T_2(w) = 2w^2 - 1$ maps the domain bounded by H_t and $H_{\pi - t}$ onto the connected component of $\mathbb{C} \setminus H_{2t}$ containing -1. Also, T_2 maps the domain bounded by the ellipse E_s onto the domain bounded by E_{2s}.*

On the basis of the above lemma, we can deduce another representation of p_k.

Theorem 4. *For $k > 0$, the function p_k is written by $p_k(z) = 1 + Q_k(\sqrt{z})^2$, where*

$$Q_k(z) = \begin{cases} \sqrt{\frac{2}{1-k^2}}\sinh(C_k \operatorname{arctanh} z) & \text{if} \quad 0 < k < 1, \\ \sqrt{\frac{1}{2\pi^2}}\operatorname{arctanh} z & \text{if} \quad k = 1, \\ \sqrt{\frac{2}{k^2-1}}\sin\left(C_k'\mathcal{K}(z/\sqrt{s}, s)\right) & \text{if} \quad 1 < k. \end{cases}$$

Here, $C_k = (2/\pi)\arccos k$ when $0 < k < 1$, and $s \in (0,1)$ is chosen so that $k = \cosh\mu(s)$ and $C_k' = (\pi/2)/\mathcal{K}(s)$ when $k > 1$.

Furthermore, the function Q_k is odd and maps the unit disk conformally onto the domain $D_k = \{x + iy : (k-1)x^2 + (k+1)y^2 < 1\}$.

Note that D_k is the inside of a hyperbola when $k < 1$ and D_k is the interior of an ellipse when $k > 1$. When $k = 1$, the domain D_k becomes the parallel strip $-1/\sqrt{2} < \operatorname{Im} z < 1/\sqrt{2}$. Also note that D_k is invariant under the involution $z \mapsto -z$.

4. Proof of the main result

In order to prove positivity of the Taylor coefficients of p_k, it is enough to show that of Q_k thanks to Theorem 4. When $0 < k < 1$, one can check that Q_k satisfies the linear differential equation

$$(1 - z^2)^2 w'' - 2z(1 - z^2)w' - C_k^2 w = 0$$

in $\mathbb{D}$. Similarly, in the case when $k > 1$, the function Q_k satisfies

$$(1 - sz^2)(1 - z^2/s)w'' - 2z((s + s^{-1})/2 - z^2)w' + \frac{C_k'^2}{s}w = 0$$

in $\mathbb{D}$, where $s \in (0,1)$ is chosen so that $k = \cosh\mu(s)$ and $C_k' = \pi/2\mathcal{K}(s)$. Note that $Q_k(z)$ satisfies $Q_k(0) = 0$ and $Q_k'(0) > 0$.

These two differential equations can be unified by the following one:

$$(1) \qquad (1 - 2Mz^2 + z^4)w'' - 2z(M - z^2)w' + cw = 0.$$

In the first case, $M = 1$ and $c = -C_k^2$ and in the second case, $M = (s + s^{-1})/2 \geq 1$ and $c = {C'_k}^2/s = \pi^2/4s\mathcal{K}(s)^2$. Furthermore, the case when $k = 1$ can be included by letting $M = 1$ and $c = 0$. At any event, one can check the inequality

$$2M - c > 0. \tag{2}$$

In fact, when $k > 1$, this is equivalent to $\mathcal{K}(s) \geq \pi/2\sqrt{1+s^2}$. However, this inequality trivially holds because

$$\mathcal{K}(s) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-s^2x^2)}} \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}.$$

Noting the above things, we now show the following lemma, which proves positivity of the Taylor coefficients of $Q_k(z)$ when $0 < k < 1$.

Lemma 5. *Assume that $M \geq 1$ and $c \leq 0$. Let $Q(z)$ be an analytic solution of (1) in $\mathbb{D}$ with $Q(0) = 0$ and $Q'(0) > 0$. Then Q has Taylor expansion in the form $Q(z) = \sum_{n=0}^\infty B_n z^{2n+1}$ and the coefficients satisfy the inequalities*

$$(2n+1)B_n - (2n-1)B_{n-1} > 0 \quad \textit{and} \quad B_n > 0 \tag{3}$$

for each $n \geq 1$.

Proof. By the form of the differential equaiton, one can easily see that a solution Q with the initial condition $Q(0) = 0$ is an odd function. Therefore, the solution can be written as above. Note that the condition $Q'(0)$ implies $B_0 > 0$. For convenience, we define $B_{-1} = 0$. Then, by the standard method, one can deduce the equation

$$(2n+2)(2n+3)B_{n+1} - \big\{2M(2n+1)^2 - c\big\}B_n + 2n(2n-1)B_{n-1} = 0 \tag{4}$$

for $n \geq 0$. The assertion is then true for $n = 0$.

We now suppose that the assertion is true up to n. Then, by (4),

$$\begin{aligned}
&(2n+2)\big\{(2n+3)B_{n+1} - (2n+1)B_n\big\} \\
=&\big\{2M(2n+1)^2 - (2n+2)(2n+1) + |c|\big\}B_n - 2n(2n-1)B_{n-1} \\
\geq&\big\{2(2n+1)^2 - (2n+2)(2n+1)\big\}B_n - 2n(2n-1)B_{n-1} \\
=&2n(2n+1)B_n - 2n(2n-1)B_{n-1} > 0
\end{aligned}$$

Therefore, the assertion is also true for $n + 1$. By induction, the proof is completed. □

When $c > 0$, we must modify the above argument. The assertion $B_n > 0$ seems to be true in the case when $M = (s + s^{-1})/2$ and $c = {C'_k}^2/s$ where $k = \cosh \mu(s)$. At least the inequality $B_1 > 0$ follows from (2).

References

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