A REFINED NOTION OF PERFECTNESS AND POTENTIAL THEORY

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Abstract. This short note is a summary of the forthcoming paper [9] of the author. We present a lower estimate of the Hausdorff content for a closed set with some density condition in a metric space. As an application, we give some estimate of generalized capacity for those sets.

1. Introduction

Let (X, ρ) be a complete metric space. A non-empty closed subset E of X is called *perfect* if E has no isolated points, namely, each point a of E is an accumulation point of $E \setminus \{a\}$. A continuum and a Cantor set are typical examples of perfect sets. A non-empty closed subset E of X is called *uniformly perfect* if there exist constants $r_0 \in (0, +\infty]$ and $c \in (0, 1]$ such that $E \cap A(a, cr, r) \neq \emptyset$ for each $a \in E$ and $r \in (0, r_0)$, where

$$A(a, t, r) = \{x \in X : t \leq \rho(x, a) \leq r\}.$$

The notion of uniform perfectness first appeared in a paper [1] by Beardon and Pommerenke and was intensively studied by Pommerenke later (see [5] and [6]) in the case when X is the complex plane. For a survey of this notion, see also [7] or [8].

The following characterization is important and, in fact, was a motivation of the present investigation. In the sequel, we will use the notation

$$B(a, r) = \{x \in X : \rho(x, a) \leq r\}$$

for $a \in X$ and $r > 0$.

Theorem 1.1 (Pommerenke [5], Järvi-Vuorinen [3]). *Let X be the complex plane $\mathbb{C}$ with Euclidean metric and let E be a non-empty closed subset of X. Then the following conditions are equivalent:*

(1) *E is uniformly perfect.*
(2) *There exists a positive constant c such that* $\mathrm{Cap}(E \cap B(a, r)) \geq cr$ *for* $r \in (0, \mathrm{diam}\, E/2)$.
(3) *There exist positive constants α and c such that* $\mathcal{H}^\infty_\alpha(E \cap B(a, r)) \geq cr^\alpha$ *for* $r \in (0, \mathrm{diam}\, E/2)$.

Here Cap *and $\mathcal{H}^\infty_\alpha$ denote the logarithmic capacity and α-dimensional Hausdorff content, respectively.*

Definitions of the logarithmic capacity and Hausdorff contents will be given in the next section. Note that the above result is indeed quantitative, i.e., the constants are estimated by each other.

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What can we say for a set with weaker density condition than uniform perfectness? We will answer to this question in this note. Suppose that a non-decreasing function $\varphi : (0, r_0) \to \mathbb{R}$ is given so that

$$0 < \varphi(r) \le r, \quad 0 < r < r_0,$$

for some $r_0 \in (0, +\infty]$. We will say that a non-empty closed subset E of X is *φ-perfect* if $E \cap A_\varphi(a, r) \neq \emptyset$ for each $a \in E$ and $0 < r < \min\{r_0, \operatorname{diam} E/2\}$, where

$$A_\varphi(a, r) = \{x \in X : \varphi(r) \le \rho(x, a) \le r\}.$$

We remark that φ-perfect sets are nothing but uniformly perfect sets when $\varphi(r) = cr$ for some constant $0 < c \le 1$ and for $r_0 = +\infty$.

In Section 2, we give basic facts about generalized capacity and Hausdorff contents. Section 3 will be devoted to presentation of estimates of Hausdorff contents, from which estimates of generalized capacity will follow.

2. Hausdorff contents and Generalized capacity

Let h be a gauge function, in other words, a strictly increasing, continuous, positive function on $(0, +\infty)$ with $h(+0) = 0$. We denote by $\mathcal{H}_h$ the Hausdorff h-measure and by $\mathcal{H}_h^\infty$ the Hausdorff h-content. More precisely, for a bounded Borel set $E \subset X$ and for $t \in (0, +\infty]$, we set

$$\mathcal{H}_h^t(E) = \inf_{\substack{E \subset \cup_j U_j \\ \operatorname{diam} U_j < t}} \sum_j h(\operatorname{diam} U_j),$$

$$\mathcal{H}_h(E) = \lim_{t \to 0} \mathcal{H}_h^t(E).$$

Note that $\mathcal{H}_h^\infty(E) > 0$ if and only if $\mathcal{H}_h(E) > 0$. When $h(r) = r^\alpha$, we also write $\mathcal{H}_h^t(E) = \mathcal{H}_\alpha^t(E)$ and $\mathcal{H}_h(E) = \mathcal{H}_\alpha(E)$. The quantities $\mathcal{H}_\alpha(E)$ and $\mathcal{H}_\alpha^\infty(E)$ are called the α-dimensional Hausdorff measure of E and α-dimensional Hausdorff content of E, respectively.

The notion of generalized capacity traces back to Frostman's thèse [2] in the case when (X, ρ) is a Euclidean space. Kametani [4] treated the general case and deduced fundamental properties of generalized capacities.

Let $\Phi : (0, \infty) \to \mathbb{R}$ be a capacity kernel, namely, a continuous, strictly decreasing function with $\Phi(+0) = +\infty$. We denote by $P(E)$ the set of Borel probability measures μ on X with $\mu(E) = 1$ for a given Borel set E. The Φ-potential u_μ^Φ of $\mu \in P(X)$ is given by

$$u_\mu^\Phi(x) = \int_X \Phi(\rho(x, y)) d\mu(y), \quad x \in X.$$

Note that u_μ^Φ is lower semi-continuous on X. Set

$$V^\Phi(E) = \inf_{\mu \in P(E)} \|u_\mu^\Phi\|_\infty,$$

where

$$\|u_\mu^\Phi\|_\infty = \sup_{x \in X} u_\mu^\Phi(x).$$

The Φ-*capacity* $C^\Phi(E)$ of E is defined by

$$C^\Phi(E) = \Phi^{-1}(V^\Phi(E)).$$

One can easily check that $C^\Phi(E) \le \operatorname{diam} E$.

When $\Phi(r) = -\log r$, $\mathrm{Cap}(E) = C^\Phi(E)$ is called the *logarithmic capacity* of E. When $\Phi(r) = r^{-\alpha}$, $C^\Phi(E)$ is called the *Newton capacity* of order $\alpha > 0$ (or $(2+\alpha)$-dimensional Newton capacity).

The following generalizes a result of Erdös-Gillis (see [10, p. 66]).

Theorem 2.1 (Kametani [4]). *Suppose that X is a complete separable metric space. Let h be a gauge function and E be a compact subset of X. When $\Phi(r) = 1/h(r)$, the condition $\mathcal{H}_h(E) < \infty$ implies $C^\Phi(E) = 0$.*

We also obtain the following weaker but quantitative result:

Lemma 2.2. *Suppose that X is a complete metric space. Let h be a gauge function and E be a compact subset of X. If $\Phi(r) = 1/h(r)$,*

$$\frac{1}{V^\Phi(E)} \le \mathcal{H}_h^\infty(E).$$

An upper estimate for $V^\Phi(E)$ is also deduced in the following way when (X, ρ) is a Euclidean space.

Theorem 2.3. *Assume that $X = \mathbb{R}^n$, $\rho(x, y) = |x - y|$ and that*

$$-\int_0^{r_1} h(r)d\Phi(r) < +\infty$$

for some $r_1 > 0$. For a compact subset E of X,

$$V^\Phi(E) \le \Phi(r_0) - \frac{A_n}{\mathcal{H}_h^\infty(E)} \int_0^{r_0} h(t)d\Phi(t)$$

holds, where $r_0 = 2\operatorname{diam} E$ and A_n is a constant depending only on n.

Corollary 2.4 (Kametani [4]). *Under the same assumptions, $\mathcal{H}_h^\infty(E) > 0 \Rightarrow C^\Phi(E) > 0$.*

3. Main results

Let h be a gauge function. We define the functions ε_1 and ε_2 by the relations

$$h(\varphi(x/3)) = \frac{\exp \varepsilon_1(x)}{2} h(x)$$

and

$$h(2x) = \frac{\exp \varepsilon_2(x)}{2} h(x)$$

for sufficiently small $x > 0$.

For a function $\lambda : (0, x_0) \to \mathbb{R}$, we say that a function $\nu : (0, x_0) \to \mathbb{R}$ is a monotone majorant of λ if ν is increasing and satisfies $|\lambda(x)| \le \nu(x)$ for $0 < x < x_0$.

Theorem 3.1. *Let (X, ρ) be a complete metric space. Suppose that φ satisfies $\varphi(r) \le cr$ in $0 < r < r_0$ for some constant $0 < c < 1$. If there is a monotone majorant ω_1 of ε_1 such that $\int_0^{r_0} \frac{\omega_1(r)dr}{r} < +\infty$, then*

$$\mathcal{H}_h^\infty(E) \ge \frac{h(\delta_0)}{2} \exp\left(-\omega_1(\delta_0) - \frac{1}{\log(6/c)} \int_0^{\delta_0} \frac{\omega_1(x)dx}{x}\right)$$

holds for any φ-perfect subset E of X, where δ_0 is an arbitrary number satisfying $0 < \delta_0 < \min\{r_0, \operatorname{diam} E/2\}$. In particular, $\mathcal{H}_h^\infty(E) > 0$. In addition, if $c < 1/4$ and if there is a *monotone majorant* ω_2 *of* ε_2 *such that* $\int_0^{r_0} \frac{\omega_2(r)dr}{r} < +\infty$, *then for each* $d_0 \in (0, r_0)$ *there exists a* φ*-perfect compact set* $E \subset [0, d_0]$ *such that*

$$\mathcal{H}_h(E) \le h(d_0) \exp\left(\omega_2(d_0) + \frac{1}{\log(1/2c)} \int_0^{d_0} \frac{\omega_2(x)dx}{x}\right),$$

and thus $0 < \mathcal{H}_h^\infty(E) \le \mathcal{H}_h(E) < +\infty$.

When X is a Euclidean space, by Theorem 2.3, we may deduce a lower estimate of generalized capacities for φ-perfects sets, though we do not state it explicitly here.

Corollary 3.2. *Let* (X, ρ) *be a complete metric space. If* $\varphi(r) = cr$, $0 < r < r_0$ *for some constants* $c \in (0, 1]$ *and* $r_0 > 0$, *then any* φ*-perfect set* E *satisfies* H-dim$E \ge \log 2/\log(6/c)$.

This fact was first shown by Järvi and Vuorinen [3] when $X = \mathbb{R}^n$ with a lower bound for the Hausdorff dimension depending on the dimension n.

We cannot apply the above theorem to the case when $\varphi(r) = ct^\alpha$ for constants $c > 0$ and $\alpha > 1$ because the Dini-type condition is not satisfied. The following results cover this case.

Theorem 3.3. *Let* (X, ρ) *be a complete metric space. Suppose that* φ *satisfies* $\varphi(r) \le cr^\alpha$ *in* $0 < r < r_0$ *for some constants* $c > 0$ *and* $\alpha > 1$. *If there is a monotone majorant* ω_1 *of* ε_1 *such that* $\int_0^{r_0} \frac{\omega_1(r)dr}{r\log(2r_0/r)} < +\infty$, *then*

$$\mathcal{H}_h^\infty(E) \ge \frac{h(\delta_0)}{2} \exp\left(-\omega_1(\delta_0) - \frac{1}{\log\alpha} \int_0^{\delta_0} \frac{\omega_1(x)dx}{x\log(M/x)}\right)$$

holds for any φ*-perfect subset* E *of* X, *where* δ_0 *is an arbitrary number satisfying* $c\delta_0^{\alpha-1} \le 1$ *and* $0 < \delta_0 < \min\{r_0, \operatorname{diam} E\}$ *and*

$$M = \left(\frac{2 \cdot 3^\alpha}{c}\right)^{1/(\alpha-1)} (> \delta_0).$$

In particular, $\mathcal{H}_h^\infty(E) > 0$.

Theorem 3.4. *Under the same assumptions, if, in addition, there is a monotone majorant* ω_2 *of* ε_2 *such that* $\int_0^{r_0} \frac{\omega_2(r)dr}{r\log(2r_0/r)} < +\infty$, *then for each* $d_0 \in (0, r_0)$ *with* $cd_0^{\alpha-1} < 1/4$ *there exists a* φ*-perfect compact set* $E \subset [0, d_0]$ *such that*

$$\mathcal{H}_h(E) \le h(d_0) \exp\left(\omega_2(d_0) + \frac{1}{\log\alpha} \int_0^{d_0} \frac{\omega_2(x)dx}{x\log(M/x)}\right)$$

and thus $0 < \mathcal{H}_h^\infty(E) \le \mathcal{H}_h(E) < +\infty$.

If we consider the particular choice $\varphi(r) = cr^\alpha$ with $\alpha > 1$, we obtain the following result as an immediate consequence.

Theorem 3.5. *Let* (X, ρ) *be a complete metric space. Let* $\varphi(r) = cr^\alpha, 0 < r < r_0$, *for constants* $c > 0, \alpha > 1, r_0 > 0$ *with* $cr_0^\alpha \le r_0$ *and let* $h(t) = (\log(2r_0/t))^{-\gamma}$, $0 < t < r_0$, *where* $\gamma = \frac{\log 2}{\log\alpha}$. *Then, every* φ*-perfect compact set* $E \subset X$ *satisfies* $\mathcal{H}_h^\infty(E) > 0$. *Furthermore, there is a compact set* $E \subset \mathbb{R}$ *such that* $0 < \mathcal{H}_h^\infty(E) \le \mathcal{H}_h(E) < \infty$.

Proof. Theorems 3.3 and 3.4 are now applicable. Thus we obtain the required assertion. □

By using Theorem 2.3 and the Erdös-Gillis theorem (Theorem 2.1 in the case when $h(r) = 1/\log(1/r)$ for r small enough), we have the following.

Corollary 3.6. *Under the same assumptions, every φ-perfect set in a Euclidean space has positive logarithmic capacity whenever $1 < \alpha < 2$. On the other hand, when $\alpha \geq 2$, there is a compact φ-perfect subset E of $\mathbb{R}$ of logarithmic capacity zero.*

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