A Refined Notion of Perfectness and Potential Theory

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Setup and uniform perfectness.

Let (X, ρ) be a complete metric space. In what follows, otherwise stated, the space X is always assumed to be a complete separable metric space, although some results may be valid for more general spaces.

A non-empty closed subset E of X is called **perfect** if E has no isolated points, namely, each point a is an accumulation point of $E\backslash\{a\}$.

Typical examples are continua or Cantor sets in the Euclidean space $\mathbb{R}^n$.

Uniform perfectness.

A non-empty closed subset E of X is called **uniformly perfect** if $\exists r_0 \in (0, +\infty]$, $\exists c \in (0, 1]$ s.t. $E \cap A(a, cr, r) \neq \emptyset$ for each $a \in E$ and $r \in (0, r_0)$, where

$$A(a, t, r) = \{x \in X : t \leq \rho(x, a) \leq r\}.$$

This notion first appeared in a paper by Beardon and Pommerenke in 1978 and was investigated more by Pommerenke later.

We will use the notation

$$B(a, r) = \{x \in X : \rho(x, a) \leq r\}.$$

Characterizations.

Theorem 1 (Pommerenke, Järvi-Vuorinen) *Let $X = \mathbb{R}^2 = \mathbb{C}$ and let E be a non-empty closed subset of X. TFAE.*

1. *E is uniformly perfect.*

2. *$\exists r_0 > 0, \exists c > 0$ s.t.*

   $$\text{Cap}(E \cap B(a,r)) \geq cr \qquad \text{for } \forall r \in (0, r_0).$$

3. *$\exists r_0 > 0, \exists \alpha > 0, \exists c > 0$ s.t.*

   $$\mathcal{H}_\alpha^\infty(E \cap B(a,r)) \geq cr^\alpha \qquad \text{for } \forall r \in (0, r_0).$$

Here Cap *and $\mathcal{H}_\alpha^\infty$ mean the logarithmic capacity and α-dimensional Hausdorff content, respectively.*

Note: This theorem is quantitative, i.e., the constants are estimated by each other. Some are generalized to the case $X = \mathbb{R}^n$, $n \geq 2$.

A more refined notion of perfectness.

Suppose that a non-decreasing function $\varphi : (0, r_0) \to \mathbb{R}$ is given so that

$$0 < \varphi(r) \leq r, \quad 0 < \forall r < r_0,$$

for some $r_0 \in (0, +\infty]$. We will say that a non-empty closed subset E of X is **φ-perfect** if $E \cap A_\varphi(a, r) \neq \emptyset$ for each $a \in E$ and $0 < r < \min\{r_0, \operatorname{diam} E/2\}$, where

$$A_\varphi(a, r) = \{x \in X; \varphi(r) \leq \rho(x, a) \leq r\}.$$

Note: φ-perfect sets are nothing but uniformly perfect sets when $\varphi(r) = cr$ for some $0 < c \leq 1$.

Hausdorff contents and measures.

Let h be a gauge function, in other words, a strictly increasing, continuous, positive function on $(0, +\infty)$ with $h(+0) = 0$. We denote by $\mathcal{H}_h$ the Hausdorff h-measure and by $\mathcal{H}_h^\infty$ the Hausdorff h-content.

More precisely, for a bounded Borel set $E \subset X$ and for $t \in (0, +\infty]$, we set

$$\mathcal{H}_h^t(E) = \inf_{\substack{E \subset \cup_j U_j \\ \operatorname{diam} U_j < t}} \sum_j h(\operatorname{diam} U_j),$$

$$\mathcal{H}_h(E) = \lim_{t \to 0} \mathcal{H}_h^t(E).$$

Note that $\mathcal{H}_h^\infty(E) > 0$ if and only if $\mathcal{H}_h(E) > 0$.

When $h(r) = r^\alpha$, we also write $\mathcal{H}_h^t(E) = \mathcal{H}_\alpha^t(E)$.

Generalized capacity.

This notion goes back to Frostman's thèse (1935) in the case when (X, ρ) is a Euclidean space. Kametani (1945) treats the general case and deduce fundamental properties of generalized capacities.

$\Phi : (0, \infty) \to \mathbb{R}$: a capacity kernel, namely, a continuous, strictly decreasing function with $\Phi(+0) = +\infty$.

$P(E)$: the set of Borel probability measures μ on X with $\mu(E) = 1$ for a Borel set E.

u_μ^Φ : the Φ-potential of $\mu \in P(X)$ given by

$$u_\mu^\Phi(x) = \int_X \Phi(\rho(x, y)) d\mu(y), \quad x \in X.$$

Note: u_μ^Φ is lower semi-continuous on X.

Set

$$V^{\Phi}(E) = \inf_{\mu \in P(E)} \|u_\mu^{\Phi}\|_\infty,$$

where

$$\|u_\mu^{\Phi}\|_\infty = \sup_{x \in X} u_\mu^{\Phi}(x).$$

$C^{\Phi}(E)$: the **Φ-capacity** of E defined by

$$C^{\Phi}(E) = \Phi^{-1}(V^{\Phi}(E)).$$

Note that $C^{\Phi}(E) \leq \operatorname{diam} E$.

When $\Phi(r) = -\log r$, $\mathrm{Cap}(E) = C^{\Phi}(E)$ is called the logarithmic capacity of E. When $\Phi(r) = r^{-\alpha}$, $C^{\Phi}(E)$ is called the Newton capacity of order $\alpha > 0$ (or $(2+\alpha)$-dimensional Newton capacity).

Relation between Hausdorff contents and generalized capacities.

The following generalizes a result of Erdös-Gillis.

Theorem 2 (Kametani (1945)) *Suppose that X is a complete separable metric space. Let h be a gauge function and E be a compact subset of X. When $\Phi(r) = 1/h(r)$, the condition $\mathcal{H}_h(E) < \infty$ implies $C^{\Phi}(E) = 0$.*

We also obtain the following weaker but quantitative result:

Lemma 3 *Suppose that X is a complete metric space. Let h be a gauge function and E be a compact subset of X. If $\Phi(r) = 1/h(r)$,*

$$\frac{1}{V^{\Phi}(E)} \leq \mathcal{H}_h^{\infty}(E).$$

An upper estimate for $V^\Phi(E)$ is also deduced in the following way when (X,ρ) is Euclidean.

Theorem 4 *Assume that* $X=\mathbb{R}^n$, $\rho(x,y)=|x-y|$ *and that*

$$-\int_0^{r_1} h(r)d\Phi(r)<+\infty$$

for some $r_1>0$. *For a compact subset* E *of* X,

$$V^\Phi(E)\le \Phi(r_0)-\frac{A_n}{\mathcal{H}_h^\infty(E)}\int_0^{r_0} h(t)d\Phi(t)$$

holds, where $r_0=2\operatorname{diam}E$ *and* A_n *is a constant depending only on* n.

Corollary 5 (Kametani) *Under the same assumptions,* $\mathcal{H}_h^\infty(E)>0\Rightarrow C^\Phi(E)>0$.

Main Theorem 1.

Let $h : (0, r_0) \to (0, +\infty)$ be a gauge function. We define the functions ε_j by the relations

$$h(\varphi(x/3)) = \frac{\exp \varepsilon_1(x)}{2} h(x)$$

and

$$h(2x) = \frac{\exp \varepsilon_2(x)}{2} h(x)$$

for sufficiently small x.

For a function $\lambda : (0, x_0) \to \mathbb{R}$, we say that a function $\nu : (0, x_0) \to \mathbb{R}$ is a monotone majorant of λ if ν is increasing and satisfies $|\lambda(x)| \leq \nu(x)$ for $0 < x < x_0$.

Theorem 6 *Let (X,ρ) be a complete metric space. Suppose that φ satisfies $\varphi(r) \le cr$ in $0 < r < r_0$ for some constant $0 < c < 1$. If there is a monotone majorant ω_1 of ε_1 such that $\int_0^{r_0} \frac{\omega_1(r)dr}{r} < +\infty$, then*

$$\mathcal{H}_h^\infty(E) \ge \frac{h(\delta_0)}{2} \exp\left(-\omega_1(\delta_0) - \frac{1}{\log(6/c)} \int_0^{\delta_0} \frac{\omega_1(x)dx}{x}\right)$$

holds for any φ-perfect subset E of X, where δ_0 is an arbitrary number satisfying $0 < \delta_0 < \min\{r_0, \operatorname{diam} E/2\}$. In particular, $\mathcal{H}_h^\infty(E) > 0$. In addition, if $c < 1/4$ and if there is a monotone majorant ω_2 of ε_2 such that $\int_0^{r_0} \frac{\omega_2(r)dr}{r} < +\infty$, then for each $d_0 \in (0, r_0)$ there exists a φ-perfect compact set $E \subset [0, d_0]$ such that

$$\mathcal{H}_h(E) \le h(d_0) \exp\left(\omega_2(d_0) + \frac{1}{\log(1/2c)} \int_0^{d_0} \frac{\omega_2(x)dx}{x}\right),$$

and thus $0 < \mathcal{H}_h^\infty(E) \le \mathcal{H}_h(E) < +\infty$.

Remark 1: When X is a Euclidean space, by Theorem 4, we may deduce a lower estimate of generalized capacities for φ-perfects sets.

Corollary 7 *Let* (X, ρ) *be a complete metric space. If* $\varphi(r) = cr,\ 0 < r < r_0$ *for some constants* $c \in (0, 1]$ *and* $r_0 > 0$, *then any* φ*-perfect set* E *satisfies* H-dim$E \geq \log 2/\log(6/c)$.

Remark 2: This fact was first shown by Järvi-Vuorinen (1996) when $X = \mathbb{R}^n$ with a lower bound depending on the dimension n.

Remark 3: We cannot apply the above theorem to the case when $\varphi(r) = ct^{\beta}$ for constants $c > 0$ and $\beta > 0$ because the Dini-type condition is not satisfied.

Main Theorem 2.

Theorem 8 *Let* (X,ρ) *be a complete metric space. Suppose that* φ *satisfies* $\varphi(r) \le cr^{\alpha}$ *in* $0 < r < r_0$ *for some constants* $c > 0$ *and* $\alpha > 1$. *If there is a monotone majorant* ω_1 *of* ε_1 *such that* $\int_0^{r_0} \frac{\omega_1(r)dr}{r \log(2r_0/r)} < +\infty$, *then*

$$\mathcal{H}_h^\infty(E) \ge \frac{h(\delta_0)}{2} \exp\left(-\omega_1(\delta_0) - \frac{1}{\log\alpha}\int_0^{\delta_0} \frac{\omega_1(x)dx}{x\log(M/x)}\right)$$

holds for any φ*-perfect subset* E *of* X, *where* δ_0 *is an arbitrary number satisfying* $c\delta_0^{\alpha-1} \le 1$ *and* $0 < \delta_0 < \min\{r_0, \operatorname{diam} E\}$ *and*

$$M = \left(\frac{2\cdot 3^{\alpha}}{c}\right)^{1/(\alpha-1)} \quad (> \delta_0).$$

In particular, $\mathcal{H}_h^\infty(E) > 0$.

Theorem 9 *Under the same assumptions, if, in addition, there is a monotone majorant* ω_2 *of* ε_2 *such that* $\int_0^{r_0} \frac{\omega_2(r)dr}{r \log(2r_0/r)} < +\infty$, *then for each* $d_0 \in (0, r_0)$ *with* $cd_0^{\alpha-1} < 1/4$ *there exists a* φ*-perfect compact set* $E \subset [0, d_0]$ *such that*

$$\mathcal{H}_h(E) \leq h(d_0) \exp\left(\omega_2(d_0) + \frac{1}{\log \alpha} \int_0^{d_0} \frac{\omega_2(x)dx}{x \log(M/x)}\right)$$

and thus $0 < \mathcal{H}_h^\infty(E) \leq \mathcal{H}_h(E) < +\infty$.

An application.

Set $\varphi(r) = cr^{\alpha}$ $(0 < r < r_0)$ for constants $c > 0, \alpha > 1, r_0 > 0$ with $cr_0^{\alpha} \leq r_0$. For this, we take $h(t) = (\log(2r_0/t))^{-\gamma},\ 0 < t < r_0$, where $\gamma = \frac{\log 2}{\log \alpha}$. Then Theorems 8 and 9 are now applicable. Thus, as a corollary, we obtain the following.

Theorem 10 *Let (X, ρ) be a complete metric space. Under the above assumptions, every φ-perfect compact set $E \subset X$ satisfies $\mathcal{H}_h^{\infty}(E) > 0$. Furthermore, there is a compact set $E \subset \mathbb{R}$ such that $0 < \mathcal{H}_h^{\infty}(E) \leq \mathcal{H}_h(E) < \infty$.*

By using Erdös-Gillis theorem, we have

Corollary 11 *Under the same assumptions, every φ-perfect set has positive logarithmic capacity whenever $1 < \alpha < 2$. On the other hand, when $\alpha \geq 2$, there is a compact φ-perfect subset E of $\mathbb{R}$ of logarithmic capacity zero.*