A Refined Notion of Perfectness and Potential Theory

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Setup and uniform perfectness.

Let \((X, \rho)\) be a complete metric space. In what follows, otherwise stated, the space \(X\) is always assumed to be a complete separable metric space, although some results may be valid for more general spaces.

A non-empty closed subset \(E\) of \(X\) is called **perfect** if \(E\) has no isolated points, namely, each point \(a\) is an accumulation point of \(E \setminus \{a\}\).

Typical examples are continua or Cantor sets in the Euclidean space \(\mathbb{R}^n\).
Uniform perfectness.

A non-empty closed subset $E$ of $X$ is called **uniformly perfect** if $\exists r_0 \in (0, +\infty]$, $\exists c \in (0, 1]$ s.t. $E \cap A(a, cr, r) \neq \emptyset$ for each $a \in E$ and $r \in (0, r_0)$, where

$$A(a, t, r) = \{x \in X : t \leq \rho(x, a) \leq r\}.$$  

This notion first appeared in a paper by Bardon and Pommerenke in 1978 and was investigated more by Pommerenke later.

We will use the notation

$$B(a, r) = \{x \in X : \rho(x, a) \leq r\}.$$
Characterizations.

Theorem 1 (Pommerenke, Järvi-Vuorinen)

Let $X = \mathbb{R}^2 = \mathbb{C}$ and let $E$ be a non-empty closed subset of $X$. TFAE.

1. $E$ is uniformly perfect.

2. $\exists r_0 > 0, \exists c > 0$ s.t.
   \[
   \text{Cap}(E \cap B(a, r)) \geq cr \quad \text{for } \forall r \in (0, r_0).
   \]

3. $\exists r_0 > 0, \exists \alpha > 0, \exists c > 0$ s.t.
   \[
   \mathcal{H}_\alpha^\infty(E \cap B(a, r)) \geq cr^\alpha \quad \text{for } \forall r \in (0, r_0).
   \]

Here $\text{Cap}$ and $\mathcal{H}_\alpha^\infty$ mean the logarithmic capacity and $\alpha$-dimensional Hausdorff content, respectively.

Note: This theorem is quantitative, i.e., the constants are estimated by each other. Some are generalized to the case $X = \mathbb{R}^n$, $n \geq 2$. 
A more refined notion of perfectness.

Suppose that a non-decreasing function \( \varphi : (0, r_0) \to \mathbb{R} \) is given so that

\[
0 < \varphi(r) \leq r, \quad 0 < \forall r < r_0,
\]

for some \( r_0 \in (0, +\infty] \). We will say that a non-empty closed subset \( E \) of \( X \) is \( \varphi \)-perfect if \( E \cap A_{\varphi}(a, r) \neq \emptyset \) for each \( a \in E \) and \( 0 < r < \min\{r_0, \text{diam } E/2\} \), where

\[
A_{\varphi}(a, r) = \{x \in X; \varphi(r) \leq \rho(x, a) \leq r\}.
\]

Note: \( \varphi \)-perfect sets are nothing but uniformly perfect sets when \( \varphi(r) = cr \) for some \( 0 < c \leq 1 \).
Hausdorff contents and measures.

Let $h$ be a gauge function, in other words, a strictly increasing, continuous, positive function on $(0, +\infty)$ with $h(0^+) = 0$. We denote by $\mathcal{H}_h$ the Hausdorff $h$-measure and by $\mathcal{H}^\infty_h$ the Hausdorff $h$-content.

More precisely, for a bounded Borel set $E \subset X$ and for $t \in (0, +\infty]$, we set

$$\mathcal{H}_h^t(E) = \inf_{E \cup \bigcup_j U_j} \sum_{\text{diam } U_j < t} h(\text{diam } U_j),$$

$$\mathcal{H}_h(E) = \lim_{t \to 0} \mathcal{H}_h^t(E).$$

Note that $\mathcal{H}^\infty_h(E) > 0$ if and only if $\mathcal{H}_h(E') > 0$.

When $h(r) = r^\alpha$, we also write $\mathcal{H}_h^t(E) = \mathcal{H}_\alpha^t(E)$. 
Generalized capacity.

This notion goes back to Frostman’s thèse (1935) in the case when \((X, \rho)\) is a Euclidean space. Kametani (1945) treats the general case and deduce fundamental properties of generalized capacities.

\[\Phi : (0, \infty) \to \mathbb{R} : \text{a capacity kernel, namely, a continuous, strictly decreasing function with } \Phi(+0) = +\infty.\]

\[P(E) : \text{the set of Borel probability measures } \mu \text{ on } X \text{ with } \mu(E) = 1 \text{ for a Borel set } E.\]

\[u_\mu^\Phi : \text{the } \Phi\text{-potential of } \mu \in P(X) \text{ given by }\]

\[u_\mu^\Phi(x) = \int_X \Phi(\rho(x, y))d\mu(y), \quad x \in X.\]

Note: \(u_\mu^\Phi\) is lower semi-continuous on \(X\).
Set
\[ V^\Phi(E) = \inf_{\mu \in P(E)} \| u^\Phi_{\mu} \|_{\infty}, \]
where
\[ \| u^\Phi_{\mu} \|_{\infty} = \sup_{x \in X} u^\Phi_{\mu}(x). \]

\( C^\Phi(E) \) : the \textbf{\( \Phi \)-capacity} of \( E \) defined by
\[ C^\Phi(E) = \Phi^{-1}(V^\Phi(E)). \]

Note that \( C^\Phi(E) \leq \text{diam } E \).

When \( \Phi(r) = -\log r \), \( \text{Cap}(E) = C^\Phi(E) \) is called the logarithmic capacity of \( E \). When \( \Phi(r) = r^{-\alpha} \), \( C^\Phi(E) \) is called the Newton capacity of order \( \alpha > 0 \) (or \((2 + \alpha)\)-dimensional Newton capacity).
Relation between Hausdorff contents and generalized capacities.

The following generalizes a result of Erdös-Gillis.

**Theorem 2 (Kametani (1945))** Suppose that $X$ is a complete separable metric space. Let $h$ be a gauge function and $E$ be a compact subset of $X$. When $\Phi(r) = 1/h(r)$, the condition $\mathcal{H}_h(E) < \infty$ implies $C^{\Phi}(E) = 0$.

We also obtain the following weaker but quantitative result:

**Lemma 3** Suppose that $X$ is a complete metric space. Let $h$ be a gauge function and $E$ be a compact subset of $X$. If $\Phi(r) = 1/h(r)$,

$$\frac{1}{V^{\Phi}(E)} \leq \mathcal{H}_h^{\infty}(E).$$
An upper estimate for $V^\Phi(E)$ is also deduced in the following way when $(X, \rho)$ is Euclidean.

**Theorem 4** Assume that $X = \mathbb{R}^n$, $\rho(x, y) = |x - y|$ and that

$$- \int_0^{r_1} h(r)d\Phi(r) < +\infty$$

for some $r_1 > 0$. For a compact subset $E$ of $X$,

$$V^\Phi(E) \leq \Phi(r_0) - \frac{A_n}{\mathcal{H}_h^\infty(E)} \int_0^{r_0} h(t)d\Phi(t)$$

holds, where $r_0 = 2\text{diam } E$ and $A_n$ is a constant depending only on $n$.

**Corollary 5 (Kametani)** Under the same assumptions, $\mathcal{H}_h^\infty(E) > 0 \Rightarrow C^\Phi(E) > 0$. 
Main Theorem 1.

Let \( h : (0, r_0) \to (0, +\infty) \) be a gauge function. We define the functions \( \varepsilon_j \) by the relations

\[
h(\varphi(x/3)) = \frac{\exp \varepsilon_1(x)}{2} h(x)
\]

and

\[
h(2x) = \frac{\exp \varepsilon_2(x)}{2} h(x)
\]

for sufficiently small \( x \).

For a function \( \lambda : (0, x_0) \to \mathbb{R} \), we say that a function \( \nu : (0, x_0) \to \mathbb{R} \) is a monotone majorant of \( \lambda \) if \( \nu \) is increasing and satisfies \( |\lambda(x)| \leq \nu(x) \) for \( 0 < x < x_0 \).
Theorem 6 Let \((X, \rho)\) be a complete metric space. Suppose that \(\varphi\) satisfies \(\varphi(r) \leq cr\) in \(0 < r < r_0\) for some constant \(0 < c < 1\). If there is a monotone majorant \(\omega_1\) of \(\varepsilon_1\) such that \(\int_0^{r_0} \frac{\omega_1(r)dr}{r} < +\infty\), then

\[
\mathcal{H}_h^\infty(E) \geq \frac{h(\delta_0)}{2} \exp \left( -\omega_1(\delta_0) - \frac{1}{\log(6/c)} \int_0^{\delta_0} \frac{\omega_1(x)dx}{x} \right)
\]

holds for any \(\varphi\)-perfect subset \(E\) of \(X\), where \(\delta_0\) is an arbitrary number satisfying \(0 < \delta_0 < \min\{r_0, \text{diam } E/2\}\). In particular, \(\mathcal{H}_h^\infty(E) > 0\). In addition, if \(c < 1/4\) and if there is a monotone majorant \(\omega_2\) of \(\varepsilon_2\) such that \(\int_0^{r_0} \frac{\omega_2(r)dr}{r} < +\infty\), then for each \(d_0 \in (0, r_0)\) there exists a \(\varphi\)-perfect compact set \(E \subset [0, d_0]\) such that

\[
\mathcal{H}_h(E) \leq \frac{h(d_0) \exp \left( \omega_2(d_0) + \frac{1}{\log(1/2c)} \int_0^{d_0} \frac{\omega_2(x)dx}{x} \right)},
\]

and thus \(0 < \mathcal{H}_h^\infty(E) \leq \mathcal{H}_h(E) < +\infty\).
Remark 1: When $X$ is a Euclidean space, by Theorem 4, we may deduce a lower estimate of generalized capacities for $\varphi$-perfects sets.

**Corollary 7** Let $(X, \rho)$ be a complete metric space. If $\varphi(r) = cr$, $0 < r < r_0$ for some constants $c \in (0, 1]$ and $r_0 > 0$, then any $\varphi$-perfect set $E$ satisfies $H\dim E \geq \log 2 / \log(6/c)$.

Remark 2: This fact was first shown by Järvi-Vuorinen (1996) when $X = \mathbb{R}^n$ with a lower bound depending on the dimension $n$.

Remark 3: We cannot apply the above theorem to the case when $\varphi(r) = ct^\beta$ for constants $c > 0$ and $\beta > 0$ because the Dini-type condition is not satisfied.
Main Theorem 2.

**Theorem 8** Let \((X, \rho)\) be a complete metric space. Suppose that \(\varphi\) satisfies \(\varphi(r) \leq cr^\alpha\) in \(0 < r < r_0\) for some constants \(c > 0\) and \(\alpha > 1\). If there is a monotone majorant \(\omega_1\) of \(\varepsilon_1\) such that \(\int_0^{r_0} \frac{\omega_1(r)dr}{r \log(2r_0/r)} < +\infty\), then

\[
\mathcal{H}^\infty_h(E) \geq \frac{h(\delta_0)}{2} \exp\left( -\omega_1(\delta_0) - \frac{1}{\log \alpha} \int_0^{\delta_0} \frac{\omega_1(x)dx}{x \log(M/x)} \right)
\]

holds for any \(\varphi\)-perfect subset \(E\) of \(X\), where \(\delta_0\) is an arbitrary number satisfying \(c\delta_0^{\alpha-1} \leq 1\) and \(0 < \delta_0 < \min\{r_0, \text{diam } E\}\) and

\[
M = \left( \frac{2 \cdot 3^\alpha}{c} \right)^{1/(\alpha-1)} (\geq \delta_0).
\]

In particular, \(\mathcal{H}^\infty_h(E) > 0\).
Theorem 9 Under the same assumptions, if, in addition, there is a monotone majorant $\omega_2$ of $\varepsilon_2$ such that $\int_0^{r_0} \frac{\omega_2(r)dr}{r \log(2r_0/r)} < +\infty$, then for each $d_0 \in (0, r_0)$ with $cd_0^{\alpha-1} < 1/4$ there exists a $\varphi$-perfect compact set $E \subset [0, d_0]$ such that

$$\mathcal{H}_h(E) \leq h(d_0) \exp \left( \omega_2(d_0) + \frac{1}{\log \alpha} \int_0^{d_0} \frac{\omega_2(x)dx}{x \log(M/x)} \right)$$

and thus $0 < \mathcal{H}_h(E) \leq \mathcal{H}_h(E) < +\infty$. 
An application.

Set \( \varphi(r) = cr^\alpha \) (\( 0 < r < r_0 \)) for constants \( c > 0, \alpha > 1, r_0 > 0 \) with \( cr_0^\alpha \leq r_0 \). For this, we take \( h(t) = (\log(2r_0/t))^{-\gamma} \), \( 0 < t < r_0 \), where \( \gamma = \frac{\log 2}{\log \alpha} \). Then Theorems 8 and 9 are now applicable. Thus, as a corollary, we obtain the following.

**Theorem 10** Let \((X, \rho)\) be a complete metric space. Under the above assumptions, every \( \varphi \)-perfect compact set \( E \subset X \) satisfies \( \mathcal{H}_h^\varphi(E) > 0 \). Furthermore, there is a compact set \( E \subset \mathbb{R} \) such that \( 0 < \mathcal{H}_h^\varphi(E) \leq \mathcal{H}_h(E) < \infty \).

By using Erdös-Gillis theorem, we have

**Corollary 11** Under the same assumptions, every \( \varphi \)-perfect set has positive logarithmic capacity whenever \( 1 < \alpha < 2 \). On the other hand, when \( \alpha \geq 2 \), there is a compact \( \varphi \)-perfect subset \( E \) of \( \mathbb{R} \) of logarithmic capacity zero.