STRONG STARLIKENESS AND STRONG CONVEXITY

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Abstract. By means of the Briot-Bouquet differential subordination, we investigate geometric properties of strongly convex functions of a prescribed order and, more generally, functions in a prescribed class. We also make numerical experiments to examine our estimates. The present note compliments the authors' paper [5] by adding some results obtained by experimental computations and details of the computations.

1. Introduction

We denote by $\mathscr{A}$ the class of functions f analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = 0$ and $f'(0) = 1$. Let $\mathscr{S}$ denote the class of normalized univalent analytic functions and $\mathscr{S}(k)$ denote the subclass of it consisting of those functions which extend to k-quasiconformal mappings for $0 \le k < 1$. Let g and h be meromorphic functions in $\mathbb{D}$. We say that g is *subordinate* to h and express it by $g \prec h$ or $g(z) \prec h(z)$ if $g = h \circ \omega$ for some analytic map $\omega : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$. When h is univalent, the condition $g \prec h$ is equivalent to $g(\mathbb{D}) \subset h(\mathbb{D})$ and $g(0) = h(0)$.

It is well recognized that the quantities

$$P_f(z) = \frac{zf'(z)}{f(z)} \quad \text{and} \quad R_f(z) = 1 + \frac{zf''(z)}{f'(z)}$$

are important for investigation of geometric properties of an analytic function f on $\mathbb{D}$. The following formulae for a composite function $f \circ g$ are useful:

$$P_{f\circ g} = P_f \circ g \cdot P_g \quad \text{and} \quad R_{f\circ g} = (R_f \circ g - 1)P_g + R_g. \tag{1.1}$$

Note also that P_f and R_f are related by

$$R_f(z) = P_f(z) + \frac{zP_f'(z)}{P_f(z)} = P_f(z) + P_f^2(z), \tag{1.2}$$

where P_f^2 means the iteration P_{P_f}. For example, $f \in \mathscr{A}$ is starlike (f is univalent and $f(\mathbb{D})$ is starlike with respect to the origin) if and only if $\operatorname{Re} P_f > 0$ and $f \in \mathscr{A}$ is convex (f is univalent and $f(\mathbb{D})$ is convex) if and only if $\operatorname{Re} R_f > 0$ (see [3]). For an analytic function h in $\mathbb{D}$ with $h(0) = 1$, following Ma and Minda [6], we define the classes $\mathscr{S}^*(h)$ and $\mathscr{K}(h)$ by

$$\mathscr{S}^*(h) = \{f \in \mathscr{A} : P_f \prec h\} \quad \text{and} \quad \mathscr{K}(h) = \{f \in \mathscr{A} : R_f \prec h\}.$$

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Throughout the paper, we will use the symbol T to stand for the mapping of the unit disk onto the right-half plane which is defined by

$$T(z) = \frac{1+z}{1-z}.$$

Note that $\mathscr{S}^* = \mathscr{S}^*(T)$ and $\mathscr{K} = \mathscr{K}(T)$ are the classical classes of (normalized) starlike and convex functions, respectively. Let α be a positive real number. A function f in $\mathscr{A}$ is said to be *strongly starlike* of order α if $f \in \mathscr{S}^*(T^\alpha)$, where the branch of $T^\alpha(z) = ((1+z)/(1-z))^\alpha$ is chosen so that $T^\alpha(0) = 1$. Many geometric characterizations of the class $\mathscr{S}^*(T^\alpha)$, $0 < \alpha < 1$, are known (cf. [13]). Similarly, f in $\mathscr{A}$ is said to be *strongly convex* of order α if $f \in \mathscr{K}(T^\alpha)$. Note that $\mathscr{S}^*(T^\alpha) \subset \mathscr{S}^*(T^{\alpha'})$ and $\mathscr{K}(T^\alpha) \subset \mathscr{K}(T^{\alpha'})$ for $0 < \alpha < \alpha'$. For a constant $0 < \kappa < 1$, we set $T_\kappa(z) = T(\kappa z)$. Here are useful criteria for quasiconformal extensions.

Theorem A.

(i) $\mathscr{S}^*(T^\alpha) \subset \mathscr{S}(\sin(\pi\alpha/2))$ *for* $0 < \alpha < 1$.
(ii) $\mathscr{S}^*(T_\kappa) \subset \mathscr{S}(\kappa)$ *for* $0 < \kappa < 1$.

Relation (i) is due to Fait, Krzyż and Zygmunt [4], and (ii) is due to Brown [1] (see also [12]). Note that $\mathscr{S}^*(T_\kappa) \subset \mathscr{S}^*(T^\alpha)$ for $\alpha = (2/\pi)\arcsin(2\kappa/(1+\kappa^2))$.

Obviously, a convex function is starlike, in other words, $\mathscr{K} \subset \mathscr{S}^*$. Therefore, it is natural to consider the problem of finding the number

$$\beta^*(\alpha) = \inf\{\beta : \mathscr{K}(T^\alpha) \subset \mathscr{S}^*(T^\beta)\}$$

for each $\alpha > 0$, or, almost equivalently, finding the number

$$\alpha^*(\beta) = \sup\{\alpha : \mathscr{K}(T^\alpha) \subset \mathscr{S}^*(T^\beta)\}$$

for each $\beta > 0$. Therefore, if $\mathscr{K}(T^\alpha) \subset \mathscr{S}^*(T^\beta)$, by definition, then $\beta^*(\alpha) \le \beta$ and $\alpha \le \alpha^*(\beta)$. It is easy to observe that $\mathscr{K}(T^\alpha) \subset \mathscr{S}^*(T^{\beta^*(\alpha)})$ and $\mathscr{K}(T^{\alpha^*(\beta)}) \subset \mathscr{S}^*(T^\beta)$. In particular,

$$\alpha \le \alpha^*(\beta^*(\alpha)) \quad \text{and} \quad \beta^*(\alpha^*(\beta)) \le \beta.$$

Mocanu showed the relation $\mathscr{K}(T^\alpha) \subset \mathscr{S}^*(T^\alpha)$ for $0 < \alpha \le 2$ in [8] and improved it for $0 < \alpha < 1$ in [9] as follows. For $0 < \beta < 1$, set

$$\gamma(\beta) = \frac{2}{\pi}\arctan\left[\tan\frac{\pi\beta}{2} + \frac{\beta}{(1+\beta)^{\frac{1+\beta}{2}}(1-\beta)^{\frac{1-\beta}{2}}\cos(\pi\beta/2)}\right]$$
$$= \beta + \frac{2}{\pi}\arctan\left[\frac{\beta\cos(\pi\beta/2)}{(1+\beta)^{\frac{1+\beta}{2}}(1-\beta)^{\frac{1-\beta}{2}} + \beta\sin(\beta\pi/2)}\right]. \tag{1.3}$$

Theorem B (Mocanu). *For* $0 < \beta < 1$, *the relation* $\mathscr{K}(T^{\gamma(\beta)}) \subset \mathscr{S}^*(T^\beta)$ *holds.*

This result was re-proved later in [10] and [11]. As immediate corollaries, we have $\gamma(\beta) \le \alpha^*(\beta)$ and $\beta^*(\gamma(\beta)) \le \beta$ for $0 < \beta < 1$. It is claimed in [11] that $\gamma(\beta) = \alpha^*(\beta)$ for $0 < \beta < 1$. That seems, however, to be wrong as we see in the sequel (cf. Example 3.1).

In the following, we consider the quantities

$$\beta^*[h] = \inf\{\beta \ge 0 : \mathscr{K}(h) \subset \mathscr{S}^*(T^\beta)\}$$

and

$$\kappa^*[h] = \inf\{\kappa \geq 0 : \mathscr{K}(h) \subset \mathscr{S}^*(T_\kappa)\}$$

for an analytic function h in $\mathbb{D}$ with $h(0) = 1$. For instance, $\beta^*[T^\alpha] = \beta^*(\alpha)$. We set $\kappa^*(\alpha) = \kappa^*[T^\alpha]$. It seems that no bounds of $\kappa^*(\alpha)$ was given in the literature. The purpose of the present paper is to give a way of estimation of $\beta^*(\alpha)$ and $\kappa^*(\alpha)$. In particular, using Theorem A, we obtain quasiconformal extension criteria for the class $\mathscr{K}(T^\alpha)$, though we do not state them separately.

2. Key theorems

Our arguments will be based on results proved by Miller and Mocanu. We state it in convenient forms for the present aim. Let Ω be the complex plane with slits $\{yi : y \geq \sqrt{3}\}$ and $\{yi : y \leq -\sqrt{3}\}$. It is known that Ω is the image of the unit disk under the univalent function $T(z) + zT'(z)/T(z)$. Also let $\mathbb{H}$ denote the right half-plane $\{z : \operatorname{Re} z > 0\}$. First result is the following.

Theorem C (Miller and Mocanu [7, Theorem 3.2j]). *Let $h : \mathbb{D} \to \Omega$ be a holomorphic map with $h(0) = 1$ and let $q : \mathbb{D} \to \mathbb{H}$ be a holomorphic map with $q(0) = 1$ satisfying the equation*

$$q(z) + \frac{zq'(z)}{q(z)} = h(z), \quad z \in \mathbb{D}.$$

Suppose that either h is convex or the function $P_q(z) = zq'(z)/q(z)$ is starlike. Then q and h must be univalent. Moreover, if an analytic function p in the unit disk with $p(0) = 1$ satisfies the condition

$$p(z) + \frac{zp'(z)}{p(z)} \prec h(z),$$

then $p \prec q$.

As an important corollary, we single out the following statement.

Corollary 2.1. *Under the hypotheses in the above theorem, the inclusion relation $\mathscr{K}(h) \subset \mathscr{S}^*(q)$ holds. In particular, the relations*

$$\beta^*[h] = \sup_{z\in\mathbb{D}} |\arg q(z)| \quad \text{and} \quad \kappa^*[h] = \sup_{z\in\mathbb{D}} \left|\frac{q(z)-1}{q(z)+1}\right|$$

hold, where the branch of $\arg q$ is taken so that $\arg q(0) = 0$.

Proof. Let $f \in \mathscr{K}(h)$. By (1.2), $P_f + zP_f'/P_f = R_f \prec h$. Thus, the theorem implies $P_f \prec q$, and therefore $f \in \mathscr{S}^*(q)$. In particular, $\sup_{z\in\mathbb{D}} |\arg P_f(z)| \leq \sup_{z\in\mathbb{D}} |\arg q(z)|$. Here, equality holds when we take f so that $R_f = h$. In the same way, the last relation is shown. □

For the choice of $q = T^\beta$, we see that $P_q(z) = 2\beta z/(1-z^2)$ is starlike, and therefore, that $\mathscr{K}(h_\beta) \subset \mathscr{S}^*(T^\beta)$, where

$$h_\beta(z) = T^\beta(z) + P_{T^\beta}(z) = \left(\frac{1+z}{1-z}\right)^\beta + \frac{2\beta z}{1-z^2}. \tag{2.1}$$

Since $\inf_{0<\theta<\pi} \arg h_\beta(e^{i\theta}) = \gamma(\beta)$, we obtain the relation $T^{\gamma(\beta)} \prec h_\beta$, and hence, $\mathscr{K}(T^{\gamma(\beta)}) \subset \mathscr{S}^*(T^\beta)$. In this way, Mocanu proved Theorem B in [9]. Note that $\kappa^*[h_\beta] = 1$ for $0 < \beta < 1$.

The function q in Theorem C can be expressed in the following way. Let $f \in \mathscr{A}$ be the function satisfying $R_f = h$. Then

$$\log f'(z) = \int_0^z \frac{h(\zeta)-1}{\zeta} d\zeta,$$

and thus,

$$f(z) = \int_0^z \exp\left[\int_0^w \frac{h(\zeta)-1}{\zeta} d\zeta\right] dw = z\int_0^1 \exp\left[\int_0^{tz} \frac{h(\zeta)-1}{\zeta} d\zeta\right] dt.$$

Since $q = P_f$, we obtain

$$\frac{1}{q(z)} = \int_0^1 \exp\left[\int_z^{tz} \frac{h(\zeta)-1}{\zeta} d\zeta\right] dt. \tag{2.2}$$

We give expressions of the quantities $\beta^*(\alpha)$ and $\kappa^*(\alpha)$ for $0 < \alpha < 1$. Let q_α be the solution of the initial value problem of the differential equation:

$$q(z) + \frac{zq'(z)}{q(z)} = T(z)^\alpha, \tag{2.3}$$
$$q(0) = 1.$$

Note that q_α is analytic in $\overline{\mathbb{D}} \setminus \{\pm 1\}$. By the symmetry of the equation, the solution q_α is symmetric, namely, $\overline{q_\alpha(z)} = q_\alpha(\bar{z})$. Since T^α is convex and satisfies $\operatorname{Re} T^\alpha > 0$, Corollary 2.1 implies the following.

Proposition 2.2. *For $0 < \alpha < 1$, the following relations hold:*

$$\beta^*(\alpha) = \sup_{0<\theta<\pi} \arg q_\alpha(e^{i\theta}) \quad \text{and} \quad \kappa^*(\alpha) = \sup_{0<\theta<\pi} \left|\frac{q_\alpha(e^{i\theta})-1}{q_\alpha(e^{i\theta})+1}\right|.$$

It seems, however, to be difficult to obtain a mathematically reliable bound of $\beta^*(\alpha)$ or $\kappa^*(\alpha)$ by solving the differential equation numerically, for the equation has a singularity at the origin. Though q_α can be presented explicitly by (2.2), it is still likely to be hard to get a good bound of $\beta^*(\alpha)$ or $\kappa^*(\alpha)$. Even the inequality $\kappa^*(\alpha) < 1$ is non-trivial (see Theorem 3.5).

We now propose elementary bounds for these quantities. For $\alpha \in (0,1), u \in (0,1), v \in (0,+\infty), c \in (0,1]$, we consider the function

$$q_{\alpha,u,v,c}(z) = \frac{(1+v)u(1+cz)^\alpha + (1-u)v(1-z)^\alpha}{u(1+cz)^\alpha + v(1-z)^\alpha}.$$

We write $h_{\alpha,u,v,c}$ for the function $q + P_q$, where $q = q_{\alpha,u,v,c}$.

Then the key theorem is now stated as follows (see [5] for the proof).

Theorem 2.3. *The function $q = q_{\alpha,u,v,c}$ is univalent in $\mathbb{D}$ and the image $q(\mathbb{D})$ is a convex subdomain of the right half-plane. Moreover, $h = h_{\alpha,u,v,c}$ is univalent, and if an analytic function p in $\mathbb{D}$ with $p(0) = 1$ satisfies $p + P_p \prec h$, then $p \prec q$.*

As an immediate consequence, we obtain the following result.

Theorem 2.4. *The relation $\mathscr{K}(h_{\alpha,u,v,c}) \subset \mathscr{S}^*(q_{\alpha,u,v,c})$ holds for $\alpha, u \in (0,1), v \in (0,\infty)$ and $c \in (0,1]$.*

We now set

$$\begin{aligned}\beta(\alpha,u,v,c) &= \sup_{z\in\mathbb{D}} \frac{2}{\pi} |\arg q_{\alpha,u,v,c}(z)|,\\ \kappa(\alpha,u,v,c) &= \sup_{z\in\mathbb{D}} \left|\frac{q_{\alpha,u,v,c}(z)-1}{q_{\alpha,u,v,c}(z)+1}\right| \quad \text{and}\\ \Gamma(\alpha,u,v,c) &= \inf_{0<\theta<\pi} \frac{2}{\pi} \arg h_{\alpha,u,v,c}(e^{i\theta})\end{aligned}$$

for $\alpha, u \in (0,1), v \in (0,\infty)$ and $c \in (0,1]$. Here, the argument is taken to be the principal value. As a corollary of the last theorem, we have

Corollary 2.5. *Let $\beta = \beta(\alpha,u,v,c), \kappa = \kappa(\alpha,u,v,c)$ and $\gamma = \Gamma(\alpha,u,v,c)$ for $\alpha, u \in (0,1), v \in (0,\infty)$ and $c \in (0,1]$. Then $\mathscr{K}(T^\gamma) \subset \mathscr{S}^*(T^\beta) \cap \mathscr{S}^*(T_\kappa)$. In particular, $\beta^*(\gamma) \le \beta$ and $\kappa^*(\gamma) \le \kappa$.*

We should note that $\Gamma(\alpha,u,v,c) = 0$ if $h_{\alpha,u,v,c}(-1) > 0$. Therefore, we should choose c so that $h_{\alpha,u,v,c}(-1) \le 0$.

The following lemma was needed to prove Theorem 2.3 and it may be of independent interest.

Lemma 2.6. *Let α be a real number with $0 < \alpha < 1$ and let a, b, c, d be non-negative numbers with $ad - bc \ne 0$. If $q = (aT^\alpha + b)/(cT^\alpha + d)$, the function $zq'(z)/q(z)$ is starlike and, in particular, univalent in $\mathbb{D}$. Here, $T^\alpha(z) = ((1+z)/(1-z))^\alpha$.*

We recall also the following simple fact (cf. [2]).

Lemma 2.7. *Let $f : \mathbb{D} \to \mathbb{C}$ be a convex univalent function and Δ be an open disk contained in $\mathbb{D}$. Then $f(\Delta)$ is also convex.*

3. Supplementary computations

We examine the estimate given in the previous section. Before investigating a concrete example, we see some basic properties of the quantities defined in the previous section.

The function $q_{\alpha,u,v,1}$ can be written in the form $L \circ T^\alpha$, where L is the Möbius transformation given by

$$L(z) = \frac{(1+v)uz + (1-u)v}{uz+v},$$

which maps the right half-plane $\mathbb{H}$ onto the disk with diameter $(1-u, 1+v)$ in such a way that $L(0) = 1-u, L(1) = 1$ and $L(\infty) = 1+v$. The relation $q_{\alpha,u,v,c} \prec q_{\alpha,u,v,1}$ holds (cf. [5]).

We denote by $\Omega(\alpha,u,v)$ the image of $\mathbb{D}$ under the mapping $q_{\alpha,u,v,1}$. This is the Jordan domain symmetric with respect to the real axis, bounded by the union of two circular arcs Γ and $\overline{\Gamma}$ with common endpoints at $1-u$ and $1+v$ which form the angle $\pi\alpha$ at the endpoints. In particular, $q_{\alpha,u,v,1}$ is a convex function. Note that $\Omega(\alpha,u,v) \subset \mathbb{H}$ for $\alpha, u \in (0,1), v \in (0,\infty)$.

We recall that $q_{\alpha,u,v,c} \prec q_{\alpha,u,v,1}$ and thus $\beta(\alpha,u,v,\alpha) \le \beta(\alpha,u,v,1)$ for $0 < c \le 1$. We now compute the value of $\beta = \beta(\alpha,u,v,1)$. Let Γ denote the upper circular arc of $\partial\Omega(\alpha,u,v)$ and let ℓ be the tangent line ℓ of Γ which passes through the origin. Further let a and R be the center and the radius of the circle containing Γ. Since $2\mathrm{Re}\, a = (1-u)+(1+v)$ and $\arg(a-(1-u)) = (1-\alpha)\pi/2$, we can write $a = 1+(v-u)/2 - R\cos(\pi\alpha/2)$ and obtain $R\sin(\pi\alpha/2) = ((1+v)-(1-u))/2 = (u+v)/2$. As is well known, letting $m = \tan(\pi\beta/2)$, distance of a to ℓ can be given by $|\mathrm{Re}\, a - m\mathrm{Im}\, a|/\sqrt{1+m^2}$, which must be equal to R. Thus we have the relation

$$\left(\frac{2-u+v}{2} + mR\cos\frac{\pi\alpha}{2}\right)^2 = (1+m^2)R^2.$$

By solving this quadratic equation in m, we obtain

$$\begin{aligned}\frac{1}{m} &= \frac{2-u+v}{u+v}\cot\frac{\pi\alpha}{2} + \sqrt{\left(\frac{2-u+v}{u+v}\right)^2 - 1}\cdot\frac{1}{\sin(\pi\alpha/2)} \\ &= \frac{(2-u+v)\cos(\pi\alpha/2) + 2\sqrt{(1-u)(1+v)}}{(u+v)\sin(\pi\alpha/2)}.\end{aligned}$$

Hence,

$$\beta(\alpha,u,v,1) = \frac{2}{\pi}\cot^{-1}\left[\frac{(2-u+v)\cos(\pi\alpha/2) + 2\sqrt{(1-u)(1+v)}}{(u+v)\sin(\pi\alpha/2)}\right].$$

One can see that the quantity $\beta = \beta(\alpha,u,v,1)$ depends only on α and $M = (2-u+v)/(u+v)$ and that β decreases in M. Note that $\beta \to \alpha$ when $M \to 1$ and $\beta \to 0$ when $M \to \infty$. In particular, $0 < \beta(\alpha,u,v,1) < \alpha$. Since $q_{\alpha,u,v,c} \prec q_{\alpha,u,v,1}$, we obtain the following.

Lemma 3.1. *For $\alpha, u \in (0,1)$, $v \in (0,\infty), c \in (0,1]$, the inequality*

$$\beta(\alpha,u,v,c) \le \frac{2}{\pi}\cot^{-1}\left[\frac{(2-u+v)\cos(\pi\alpha/2) + 2\sqrt{(1-u)(1+v)}}{(u+v)\sin(\pi\alpha/2)}\right]$$

holds, where equality is valid whenever $c = 1$.

Though it is hard to give an explicit expression of $\beta(\alpha,u,v,c)$ except for the case $c=1$, the quantity $\kappa(\alpha,u,v,c)$ can easily be computed.

Lemma 3.2. *For $\alpha, u \in (0,1), v \in (0,\infty), c \in (0,1]$, the quantity $\kappa(\alpha,u,v,c)$ is given by*

$$\kappa(\alpha,u,v,c) = \max\left\{\frac{uv(1-b^\alpha)}{u(2+v)b^\alpha + (2-u)v}, \frac{v}{2+v}\right\}$$

where $b = (1-c)/2$.

Proof. Let $q = q_{\alpha,u,v,c}$ and set $h = (q-1)/(q+1)$. Our goal is to show that $h(\mathbb{D}) \subset \mathbb{D}_\kappa = \{z : |z| < \kappa\}$.

By Lemma 2.7, the image of $\mathbb{D}$ under the function $f = T^\alpha \circ \omega$ is convex, where $\omega : \mathbb{D} \to \mathbb{D}$ is given by

$$\omega(z) = \omega_c(z) = \frac{(1+c)z}{2-(1-c)z} = \frac{az}{1-bz}, \tag{3.1}$$

where $a = (1+c)/2$ and $b = (1-c)/2$. Note that $f(-1) = b^\alpha$. Since f is symmetric with respect to the real axis, $f(\mathbb{D})$ is contained in the half-plane $H = \{z : \mathrm{Re}\, z > b^\alpha\}$. Recalling that $h = M \circ f$, where M is the Möbius transformation $uv(z-1)/((2+v)uz + (2-u)v)$, we find that $h(\mathbb{D})$ is contained in the disk $M(H)$. The disk $M(H)$ has $(h(-1), h(1))$ as a diameter and therefore contained in the disk $|w| < \max\{-h(-1),\ h(1)\} = \kappa$. The proof is completed. □

Let us give a rough lower estimate for the quantity $\gamma = \Gamma(\alpha, u, v, 1)$. (Though we can give a similar, but more complicated, estimate of $\Gamma(\alpha, u, v, c)$ for $c \in (0, 1]$, we are content with the present case.) Recall that γ is defined to be the infimum of $\arg h(e^{i\theta}) = \arg(q(e^{i\theta}) + Q(e^{i\theta}))$ over the range $0 < \theta < \pi$, where $q = q_{\alpha,u,v,1},\ Q = P_q,\ h = q + Q$.

It is easy to see that $|Q(e^{i\theta})| \to \infty$ and $\arg Q(e^{i\theta}) \to (\pi/2)(1-\alpha)$ as $\theta \to +0$, and that $|Q(e^{i\theta})| \to \infty$ and $\arg Q(e^{i\theta}) \to (\pi/2)(1+\alpha)$ as $\theta \to \pi - 0$. Since Q is starlike (Lemma 2.6) and analytic in $\overline{\mathbb{D}} \setminus \{1, -1\}$, $\arg Q(e^{i\theta})$ is increasing and thus,

$$\frac{\pi}{2}(1-\alpha) < \arg Q(e^{i\theta}) < \frac{\pi}{2}(1+\alpha) \tag{3.2}$$

for $0 < \theta < \pi$. On the other hand, $q(e^{i\theta})$ is bounded. Therefore, $\arg h(e^{i\theta}) \to (\pi/2)(1-\alpha)$ as $\theta \to +0$. In particular,

$$\Gamma(\alpha, u, v, c) \le 1 - \alpha$$

when $c = 1$. It is not difficult to see that the same is true for $c \in (0, 1]$. This estimate shows a limitation of Theorem 2.3 in applications.

For a lower estimate, we set

$$\Phi(\theta) = \arg q(e^{i\theta}) \quad \text{and} \quad \Psi(\theta) = \arg Q(e^{i\theta})$$

for $0 < \theta < \pi$.

Lemma 3.3. *Let θ_0 be the number in $(0, \pi)$ satisfying $\operatorname{Re} Q(e^{i\theta_0}) = 0$. Then $\Phi'(\theta) > 0$ for $0 < \theta < \theta_0$ and $\Phi'(\theta) < 0$ for $\theta_0 < \theta < \pi$. Furthermore, $|\Phi'(\theta)| \le \Psi'(\theta)$ holds for $0 < \theta < \pi$.*

Proof. Recall that Q is starlike and analytic in $\overline{\mathbb{D}} \setminus \{1, -1\}$ and therefore θ_0 is uniquely determined. We also note that

$$\Phi'(\theta) = \frac{d}{d\theta}\left(\operatorname{Im} \log q(e^{i\theta})\right) = \operatorname{Re} Q(e^{i\theta})$$

for $0 < \theta < \pi$. Therefore, the first assertion is clear. Similarly, we have $\Psi'(\theta) = \operatorname{Re} P_Q(e^{i\theta})$. Therefore, in order to show the second assertion, we need to see

$$-\operatorname{Re} P_Q(e^{i\theta}) \le \operatorname{Re} Q(e^{i\theta}) \le \operatorname{Re} P_Q(e^{i\theta})$$

for $0 < \theta < \pi$. Since $R_q = Q + P_Q$ by (1.2), the left-hand side inequality follows from convexity of q. We show the right-hand side. For convenience, set $a = (1+v)u, b = (1-u)v, c = u$ and $d = v$ for a while. (We should forget about the previous parameter c since we are now assuming that $c = 1$.) We also set $p = T^\alpha$. Then,

$$Q(z) = \frac{zp'(z)}{p(z)} \cdot \frac{(ad-bc)p(z)}{(ap(z)+b)(cp(z)+d)} = \frac{2\alpha z}{1-z^2} \cdot \frac{(ad-bc)p(z)}{(ap(z)+b)(cp(z)+d)}, \tag{3.3}$$

and

$$\begin{aligned} P_Q(z) &= \frac{1+z^2}{1-z^2} + \frac{zp'(z)}{p(z)} - \frac{azp'(z)}{ap(z)+b} - \frac{czp'(z)}{cp(z)+d} \\ &= \frac{1+z^2}{1-z^2} + \frac{2\alpha z}{1-z^2} \cdot \frac{bd - acp(z)^2}{(ap(z)+b)(cp(z)+d)} \end{aligned} \tag{3.4}$$

Thus, we compute

$$P_Q(z) - Q(z) = \frac{1+z^2}{1-z^2} + \frac{2\alpha z}{1-z^2} \cdot \frac{b - ap(z)}{b + ap(z)},$$

where $p(z) = ((1+z)/(1-z))^\alpha$. Therefore,

$$\begin{aligned}\operatorname{Re} P_Q(e^{i\theta}) - \operatorname{Re} Q(e^{i\theta}) &= -\frac{\alpha}{\sin\theta} \cdot \operatorname{Im} \frac{b - ap(e^{i\theta})}{b + ap(e^{i\theta})} \\ &= \frac{\alpha}{\sin\theta} \cdot \frac{ab \cdot \operatorname{Im} p(e^{i\theta})}{|b + ap(e^{i\theta})|^2} > 0\end{aligned}$$

for $0 < \theta < \pi$. □

As we saw above, $\Psi(0+) = \pi(1-\alpha)/2$. Therefore,

$$\Psi(\theta) - \Phi(\theta) \geq \Psi(0+) - \Phi(0+) = \frac{\pi(1-\alpha)}{2}$$

for $0 < \theta < \pi$. Set $\rho(\theta) = |q(e^{i\theta})|$ and $R(\theta) = |Q(e^{i\theta})|$. We now use the following elementary inequality:

$$\begin{aligned}\arg(q(e^{i\varphi}) + Q(e^{i\psi})) &= \Phi(\theta) + \arg(\rho(\theta) + R(\theta)e^{i(\Psi(\theta)-\Phi(\theta))}) \\ &= \Phi(\theta) + \arcsin\left(\frac{\sin(\Psi(\theta) - \Phi(\theta))}{1 + \rho(\theta)/R(\theta)}\right) \\ &\geq \Phi(\theta) + \arcsin\left(\frac{\sin(\pi(1-\alpha)/2)}{1 + \rho(\theta)/R(\theta)}\right)\end{aligned}$$

for $0 < \theta < \pi$. Thus we have proved the following lemma.

Lemma 3.4.

$$\arg h(e^{i\theta}) \geq \Phi(\theta) + \arcsin\left(\frac{\sin(\pi(1-\alpha)/2)}{1 + \rho(\theta)/R(\theta)}\right).$$

If we set $t = \cot(\theta/2)$, we obtain the representation

$$\begin{aligned}h(e^{i\theta}) &= q(e^{i\theta}) + Q(e^{i\theta}) \\ &= \frac{(1+v)u\zeta t^\alpha + (1-u)v}{u\zeta t^\alpha + v} + \frac{\alpha uv(u+v)(t+1/t)t^\alpha \zeta i}{2((1+v)u\zeta t^\alpha + (1-u)v)(u\zeta t^\alpha + v)},\end{aligned}$$

where $\zeta = e^{\pi\alpha i/2}$. Therefore,

$$\begin{aligned}\frac{\rho(\theta)}{R(\theta)} &= \frac{2|(1+v)u\zeta t^\alpha + (1-u)v|^2}{\alpha uv(u+v)(t+1/t)t^\alpha} \\ &= \frac{2((1+v)^2u^2t^\alpha + 2(1-u)(1+v)uv\cos(\pi\alpha/2) + (1-u)^2v^2t^{-\alpha})}{\alpha uv(u+v)(t+1/t)} \qquad (3.5) \\ &\leq \frac{2(u+v)}{\alpha uv} \min\left\{t^{\alpha-1}, t^{1-\alpha}\right\}.\end{aligned}$$

Though we could obtain an explicit (but complicated) bound for $\kappa^*(\gamma)$ for small enough γ, we just state the result in the following qualitative form.

Theorem 3.5. *Let* $\gamma_0 \approx 0.576567$ *be the solution of the equation* $\gamma(1-x) = x$ *in* $0 < x < 1$, *where* $\gamma(\beta)$ *is the function given in* (1.3). *Then* $\kappa^*(\gamma) < 1$ *for* $\gamma \in (0, \gamma_0)$.

Proof. Set $\alpha = 1 - \gamma_0$. Then $\gamma(\alpha) = \gamma_0$. Fix a number $\gamma \in (0, \gamma_0)$ and choose a $\tau > 1$ so large that

$$\frac{\pi\gamma}{2} < \arcsin\left(\frac{\sin(\pi\gamma_0/2)}{1 + 8\tau^{\alpha-1}/\alpha}\right). \tag{3.6}$$

Let θ_1 and θ_2 be the angles determined by $\cot(\theta_1/2) = \tau$ and $\cot(\theta_2/2) = 1/\tau$ and $0 < \theta_1 < \theta_2 < \pi$.

It is easy to see that $h_{\alpha,u,v,1}$ converges to h_α locally uniformly on $\overline{\mathbb{D}} \setminus \{1, -1\}$ as $u \to 1$ and $v \to +\infty$, where $h_{\alpha,u,v,c}$ is the function defined in §3 and h_α is given by (2.1). We now recall the fact that $\inf_{0<\theta<\pi} \arg h_\alpha(e^{i\theta}) = \gamma(\alpha) = \gamma_0$ (cf. [9]). Therefore, we can take $u \in (1/2, 1)$ and $v \in (1, \infty)$ so that $\arg h_{\alpha,u,v,1}(e^{i\theta}) > \gamma$ for $\theta \in [\theta_1, \theta_2]$. At the same time, by Lemma 3.4, (3.5) and (3.6), we have

$$\begin{aligned}\arg h_{\alpha,u,v,1}(e^{i\theta}) &\geq \arcsin\left(\frac{\sin(\pi\gamma_0/2)}{1 + 2(u+v)\tau^{\alpha-1}/(\alpha uv)}\right)\\ &\geq \arcsin\left(\frac{\sin(\pi\gamma_0/2)}{1 + 8\tau^{\alpha-1}/\alpha}\right)\\ &> \frac{\pi\gamma}{2}\end{aligned}$$

for $\theta \in (0, \theta_1) \cup (\theta_2, \pi)$. In this way, we conclude that $\Gamma(\alpha, u, v, 1) \geq \gamma$ for this choice of (α, u, v).

On the other hand, by Lemma 3.2, we have

$$\kappa(\alpha, u, v, 1) = \min\left\{\frac{u}{2-u}, \frac{v}{2+v}\right\} < 1.$$

Therefore, we obtain $\mathscr{K}(T^\gamma) \subset \mathscr{K}(h_{\alpha,u,v,1}) \subset \mathscr{S}^*(q_{\alpha,u,v,1}) \subset \mathscr{S}^*(T_{\kappa(\alpha,u,v,1)})$, from which we deduce $\kappa^*(\gamma) \leq \kappa(\alpha, u, v, 1) < 1$. □

In the last theorem, the assumption $\gamma < \gamma_0$ was put for merely a technical reason. It is true that $\kappa^*(\gamma) < 1$ for every $\gamma \in (0, 1)$. See [5] for a rigorous proof of it.

Example 3.1. We try to estimate $\beta^*(1/2)$ with the aid of Mathematica. By numerical experiments, we found that the choice $\alpha = 0.4731, u = 0.9285, v = 4.2506, c = 0.9285$ yields $\Gamma(\alpha, u, v, c) \approx 1/2$ and $\beta(\alpha, u, v, c) \approx 0.32104$. Therefore, we obtain numerically, $\beta^*(1/2) < 0.3211$.

Mocanu's theorem, in turn, gives the estimate $\beta^*(1/2) \leq \gamma^{-1}(1/2) \approx 0.35046$. On the other hand, by numerically solving the differential equation (2.3), we obtain an experimental value $\beta^*(1/2) \approx 0.309$, though we do not know how reliable it is.

We next try to estimate $\kappa^*(1/2)$. For $\alpha = 1/2, u = 0.95, v = 3.47, c = 0.49$, we obtain $\Gamma(\alpha, u, v, c) \approx 1/2$ and $\kappa(\alpha, u, v, c) \approx 0.634$. Therefore, $\kappa^*(1/2) < 0.635$. By a numerical computation, we have an experimental value $\kappa^*(1/2) \approx 0.613$.

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