平面領域の環状幅 (CIRCULAR WIDTH) とその応用

THE CIRCULAR WIDTH OF A PLANE DOMAIN AND ITS APPLICATIONS

須川 敏幸
TOSHIYUKI SUGAWA

Abstract. We first give a quantity for non-vanishing analytic functions on a hyperbolic domain to measure their rate of growth in terms of hyperbolic metric. Then we define the circular width of a plane domain not containing the origin. We will see that an estimate of the circular width leads to a criterion for univalence (or quasiconformal extension).

1. The quantity $V_D(\varphi)$

Conformal invariants play a central role in the modern theory of functions of a complex variable. One of the most important is the hyperbolic metric $\rho_D(z)|dz|$ of a hyperbolic plane domain D. Recall that a subdomain D of $\mathbb{C}$ is called hyperbolic if D admits an analytic universal covering projection p of the unit disk $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ onto D. Then the hyperbolic metric is defined by the equation $\rho_D(z)|p'(\zeta)| = 1/(1-|\zeta|^2)$ for $\zeta \in p^{-1}(z)$. Note that the density $\rho_D(z)$ does not depend on the particular choice of ζ or p. The Poincaré-Koebe uniformization theorem tells us that $D \subset \mathbb{C}$ is hyperbolic if and only if D is neither the whole plane $\mathbb{C}$ nor the punctured plane $\mathbb{C} \setminus \{a\}$ for any $a \in \mathbb{C}$. We denote by $d_D(z_0, z_1)$ the distance between z_0 and z_1 measured by the metric $\rho_D(z)|dz|$.

Let φ be a non-vanishing analytic function on a hyperbolic domain D, namely, $\varphi : D \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is holomorphic. Then we set

$$V_D(\varphi) = \sup_{z \in D} \rho_D(z)^{-1} \left| \frac{\varphi'(z)}{\varphi(z)} \right|.$$

Note also that $V_D(\varphi)$ can be thought of the Bloch semi-norm of the (possibly multi-valued) function $\log \varphi$. The quantity $V_D(\varphi)$ measures the rate of growth of φ compared with the hyperbolic metric. More precisely, we have the following characterization.

Proposition 1. *Let φ be a non-vanishing analytic function on a hyperbolic domain D and let c be a positive constant. Then $V_D(\varphi) \le c$ if and only if the double inequality*

$$\exp\big(-c\, d_D(z_0, z_1)\big) \le \frac{|\varphi(z_1)|}{|\varphi(z_0)|} \le \exp\big(c\, d_D(z_0, z_1)\big) \tag{1.1}$$

holds for every pair of points z_0, z_1 in D.

We list properties of this quantity.

Theorem 2. *Let D be a hyperbolic domain and let φ and ψ be non-vanishing analytic functions on D.*

(i) $V_D(\varphi \cdot \psi) \le V_D(\varphi) + V_D(\psi)$.
(ii) $V_D(\varphi^\alpha) = |\alpha| V_D(\varphi)$ *holds for a complex number α whenever φ^α is single-valued in D.*
(iii) *Let $p : D_0 \to D$ be an analytic (unbranched and unlimited) covering projection. Then $V_{D_0}(\varphi \circ p) = V_D(\varphi)$. In particular, $V_D(\varphi)$ is conformally invariant in the sense that this does not depend on the source domain.*
(iv) $V_D(L \circ \varphi) = V_D(\varphi)$ *holds for any conformal automorphism L of $\mathbb{C}^*$. In particular, $V_D(1/\varphi) = V_D(\varphi) = V_D(c\varphi)$ for any constant $c \in \mathbb{C}^*$.*

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(v) *Let* $\omega : D_0 \to D$ *be a holomorphic map. Then* $V_{D_0}(\varphi \circ \omega) \le V_D(\varphi)$.
(vi) *If* $\psi : D \to \mathbb{C}^*$ *is univalent and if* $\varphi(D) \subset \psi(D)$ *then* $V_D(\varphi) \le V_D(\psi)$.

2. Circular width

Let Ω be a hyperbolic plane domain with $0 \in \mathbb{C} \setminus \Omega$. The quantity

$$W(\Omega) = \left(\inf_{w \in \Omega} |w| \rho_\Omega(w) \right)^{-1}$$

will be called the *circular width* of Ω (about the origin). This quantity can be expressed in terms of a covering projection onto Ω.

Lemma 3. *Let* Ω *be a proper subdomain of the punctured plane* $\mathbb{C}^*$ *and let* p *be an analytic (unbranched) covering projection of a domain* D *onto* Ω. *Then* $W(\Omega) = V_D(p)$.

Remark. In general, we can define the circular width $W_a(\Omega)$ about a point $a \in \mathbb{C} \setminus \Omega$ in the same way: $W_a(\Omega) = 1/\inf_{w\in\Omega} |w-a|\rho_\Omega(w)$. It is known that the domain constant $C(\Omega) = \sup_{a \in \mathbb{C}\setminus\Omega} W_a(\Omega)$ is finite if and only if the set $\widehat{\mathbb{C}} \setminus \Omega$ is uniformly perfect (see, for example, [10] or [12]). In this context, the constant $W_a(\Omega)$ appeared essentially in a paper [13] by J.-H. Zheng.

We now collect basic properties of the circular width. Before that, we recall the notion of circular symmetrization. For a subdomain Ω of $\mathbb{C}^*$ we define the circular symmetrization Ω^* (about the origin) by

$$\Omega^* = \{re^{i\theta} : \theta \in I(r, \Omega), 0 < r < \infty\},$$

where $I(r, \Omega)$ denotes the interval in the form $(-t/2, t/2)$ of the same length as $I_r = \{\theta \in [-\pi, \pi] : re^{i\theta} \in \Omega\}$ if $I_r \ne [-\pi, \pi]$ otherwise $I(r, \Omega) = [-\pi, \pi]$.

Theorem 4. *Let* Ω *and* Ω' *be proper subdomains of the punctured plane* $\mathbb{C}^*$.
(i) $W(\Omega) = W(L(\Omega))$ *for any conformal automorphism* L *of* $\mathbb{C}^*$.
(ii) *If* $\Omega \subset \Omega'$, *then* $W(\Omega) \le W(\Omega')$.
(iii) *Circular symmetrization does not decrease circular width;* $W(\Omega) \le W(\Omega^*)$.
(iv) *If* Ω *is simply connected, then* $W(\Omega) \le 4$.

In general, the circular width may not be finite. We give here a characterization of domains with infinite circular width. In particular, if the origin is an isolated boundary point of Ω, then $W(\Omega) = \infty$.

Proposition 5. *Let* Ω *be a proper subdomain of the punctured plane* $\mathbb{C}^*$. *The circular width* $W(\Omega)$ *is infinite if and only if there is a sequence of annuli* $A_n = \{w \in \mathbb{C} : r_n < |w| < R_n\}$ *with* $A_n \subset \Omega$ *such that* $R_n/r_n \to \infty$.

The circular width may not behave continuously in Ω. For instance, consider the sequence of domains $\Omega_n = \{|w-1| < 1 + 1/n\}$. Then Ω_n converges to $\Omega_\infty = \{|w-1| < 1\}$ in the Hausdorff topology. But $W(\Omega_n) = \infty$ by Proposition 5 whereas $W(\Omega_\infty) \le W(\mathbb{H}) = 2$ (see Example 3.1). We can, however, show a continuity property of circular width in the following form.

Proposition 6. *Let* Ω_n *be a sequence of domains with* $\Omega_n \subset \Omega_{n+1}$ *such that the union* $\Omega = \cup_{n=1}^\infty \Omega_n$ *is a proper subdomain of* $\mathbb{C}^*$. *Then* $W(\Omega_n) \to W(\Omega)$ *as* $n \to \infty$.

The circular width $W(\Omega)$ dominates the quantity $V_D(\varphi)$ for holomorphic maps $\varphi : D \to \Omega$.

Theorem 7. *Let* $\varphi : D \to \Omega$ *be holomorphic. Then* $V_D(\varphi) \le W(\Omega)$.

Proof. By Theorem 2 (v), we have $V_D(\varphi) = V_D(\mathrm{id}_\Omega \circ \varphi) \le V_\Omega(\mathrm{id}_\Omega) = W(\Omega)$. □

Combining this with Proposition 1, we have the following. A similar result was obtained by [13].

Corollary 8. *Under the same hypotheses in Theorem 7,*

$$\exp\big(-W(\Omega)\, d_D(z_0, z_1)\big) \le \frac{|\varphi(z_1)|}{|\varphi(z_0)|} \le \exp\big(W(\Omega)\, d_D(z_0, z_1)\big), \quad z_0, z_1 \in D.$$

3. Computations of circular widths

In the present section, we give exact values of circular width for several concrete examples. In view of Theorem 4 (iii), we see that circularly symmetric domains are particularly important.

Example 3.1 (sectors)**.** For $S(\beta)=\{w:|\arg w|<\pi\beta/2\},\ 0<\beta\le 2$, we have $W(S(\beta))=2\beta$.

Example 3.2 (half-sectors)**.** Let $S(\beta,r)=\{w:|\arg w|<\pi\beta/2,|w|<r\}$ and $S'(\beta,r)=\{w:|\arg w|<\pi\beta/2,|w|>1/r\}$ for $0<\beta\le 2$ and $0<r<\infty$. Then $W(S(\beta,r))=W(S'(\beta,r))=2\beta$.

Example 3.3 (annuli)**.** For the annulus $A(r,R)=\{w:r<|w|<R\},\ 0<r<R<\infty$, we have $W(A(r,R))=(2/\pi)\log(R/r)$.

Example 3.4 (disks)**.** Let $\mathbb{D}(a,r)=\{w:|w-a|<r\}$ for $0<r\le a$. Then

$$W(\mathbb{D}(a,r))=\frac{2r/a}{1+\sqrt{1-(r/a)^2}}.$$

Example 3.5 (parallel strips)**.** Let $P(a,b)=\{w:a<\operatorname{Re}w<b\}$ for $0\le a<b<\infty$. Then

$$W(P(a,b))=\max_{0\le\theta\le\pi/2}\frac{2t\cos\theta}{1-t\theta},$$

where t is a number with $0<t\le 2/\pi$ determined by $\pi t/2=(b-a)/(b+a)$.

Example 3.6 (truncated wedges)**.** Let $S(\beta,r,R)=\{w:|\arg w|<\pi\beta/2,r<|w|<R\},\ 0<\beta\le 2,0<r<R<\infty$. Then

$$W(\Omega)=\frac{\log(R/r)}{(1+t)\mathbf{K}(t)},\quad\text{where}\quad \mathbf{K}(t)=\int_0^1\frac{dx}{\sqrt{(1-x^2)(1-t^2x^2)}}$$

is the complete elliptic integral of the first kind and $0<t<1$ is a number such that

$$\frac{\mathbf{K}(\sqrt{1-t^2})}{\mathbf{K}(t)}=\frac{2\pi\beta}{\log(R/r)}.$$

Note that the quantity $\mu(t)=(\pi/2)\mathbf{K}(\sqrt{1-t^2})/\mathbf{K}(t)$ is the modulus of the Grötzsch ring $\mathbb{D}\setminus[0,t]$ for $0<t<1$ and decreasing from $+\infty$ to 0 (see, for example, [1]). Therefore, we can always take such a t satisfying the above relation.

We remark that the essentially same observations for the last example were made by Avhadiev and Aksent'ev [2] though they did not make systematic use of circular width.

4. Applications

In this section, we give a few applications of circular width. More concrete applications can be found in [9] and [11].

Let us introduce some notation. For a locally univalent function f on $\mathbb{D}$, the quantity $T_f=f''/f'$ is called the pre-Schwarzian derivative of f and measured by the norm

$$\|T_f\|_{\mathbb{D}}=\sup_{z\in\mathbb{D}}(1-|z|^2)|T_f(z)|.$$

Note that this can be described by $\|T_f\|_{\mathbb{D}}=V_{\mathbb{D}}(f')$. Let $\mathcal{A}$ denote the class of holomorphic functions f on $\mathbb{D}$ normalized by $f(0)=0,f'(0)=1$.

Theorem 9. *Let Ω be a proper subdomain of the punctured plane $\mathbb{C}^*$ with $W(\Omega)<2$. If $f\in\mathcal{A}$ satisfies $f'(\mathbb{D})\subset\Omega$, then $|f(z)|<M,\ z\in\mathbb{D}$. Here M is a constant depending only on $W(\Omega)$.*

The constant M can be given by

$$M=\lambda\Big[\psi\big(-\tfrac{\lambda}{2}\big)-\psi\big(\tfrac{1-\lambda}{2}\big)\Big]-1,$$

where $\lambda=W(\Omega)/2$ and $\psi(x)=\Gamma'(x)/\Gamma(x)$ is the digamma function (see [7]).

It may be interesting to find a characterization of such subdomains Ω of $\mathbb{H}$ that $f'(\mathbb{D})\subset\Omega$ implies boundedness of $f\in\mathcal{A}$. Note that the condition $f'(\mathbb{D})\subset\mathbb{H}$ implies univalence of f (Noshiro-Warschawski theorem). Recently, Chuaqui and Gevirtz [6] gave a characterization of such subdomains Ω of $\mathbb{H}$ that $f'(\mathbb{D}) \subset \Omega$ implies quasiconformal extensibility of $f \in \mathcal{A}$.

We next apply Theorem 7 to the problem of quasiconformal extensibility. Our result is based the following theorem due to J. Becker. See, for sharpness, Becker and Pommerenke [5].

Theorem 10 (Becker [4])**.** *Let $f \in \mathcal{A}$ be locally univalent. If $\|T_f\|_{\mathbb{D}} \le 1$, then f is univalent. Furthermore, if $\|T_f\| \le k$ for $k \in [0,1)$, then f has a K-quasiconformal extension to the whole plane, where $K = (1+k)/(1-k)$.*

Making use of this result, we can show the following.

Theorem 11. *Suppose that a proper subdomain Ω of the punctured plane $\mathbb{C}^*$ satisfies $W(\Omega) \le k$ for some $k \le 1$. If $f'(\mathbb{D}) \subset \Omega$ for $f \in \mathcal{A}$, then f is univalent and, moreover, f has a K-quasiconformal extension to the whole plane when $K = (1+k)/(1-k) < \infty$.*

See [11] for a counterpart of the theorem for meromorphic functions.

Proof. As we noted, the condition $f'(\mathbb{D}) \subset \Omega$ implies that $\|T_f\|_{\mathbb{D}} \le W(\Omega) \le k$. We now apply Theorem 10 to deduce the assertions. □

Combining this with examples presented in the previous section, we obtain a series of corollaries. (Remember the fact that circular width is invariant under rotations.) The first corollary was noted by Avhadiev and Aksent'ev [3, pp. 33–34] at least when $\gamma = 0$.

Corollary 12. *Let $0 < k \le 1$ and $f \in \mathcal{A}$. If $|\arg f'(z) - \gamma| < \pi k/4$ in $|z| < 1$ for some real constant γ, then f is univalent and, moreover, it extends to a K-quasiconformal mapping of the whole plane when $K = (1+k)/(1-k) < \infty$.*

Note also that the condition $|\arg f'(z)| < M,\ |z| < 1$, implies quasiconformal extensibility of f when $M < \pi/2$ (see [6]).

Corollary 13. *Let k, r, R be positive numbers with $0 < \log(R/r) \le \pi k/2$ and $f \in \mathcal{A}$. If $r < |f'(z)| < M$ in $|z| < 1$, then f is univalent and, moreover, it extends to a K-quasiconformal mapping of the whole plane when $K = (1+k)/(1-k) < \infty$.*

We omit the other examples.

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Department of Mathematics, Graduate School of Science, Hiroshima University, Higashi-Hiroshima, 739-8526 Japan
E-mail address: sugawa@math.sci.hiroshima-u.ac.jp