THE CIRCULAR WIDTH OF A PLANE DOMAIN AND ITS APPLICATIONS

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Abstract. We first give a quantity for non-vanishing analytic functions on a hyperbolic domain to measure their rate of growth in terms of hyperbolic metric. Then we define the circular width of a plane domain not containing the origin. We will see that an estimate of the circular width leads to a criterion for univalence (or quasiconformal extension).

1. The quantity $V_D(\varphi)$

Conformal invariants play a central role in the modern theory of functions of a complex variable. One of the most important is the hyperbolic metric $\rho_D(z)|dz|$ of a hyperbolic plane domain $D$. Recall that a subdomain $D$ of $\mathbb{C}$ is called hyperbolic if $D$ admits an analytic universal covering projection $p$ of the unit disk $D = \{ \zeta \in \mathbb{C} : |\zeta| < 1 \}$ onto $D$. Then the hyperbolic metric is defined by the equation $\rho_D(z)|p'(\zeta)| = 1/(1-|\zeta|^2)$ for $\zeta \in p^{-1}(z)$. Note that the density $\rho_D(z)$ does not depend on the particular choice of $\zeta$ or $p$. The Poincaré-Koebe uniformization theorem tells us that $D \subset \mathbb{C}$ is hyperbolic if and only if $D$ is neither the whole plane $\mathbb{C}$ nor the punctured plane $\mathbb{C} \setminus \{a\}$ for any $a \in \mathbb{C}$. We denote by $d_D(z_0, z_1)$ the distance between $z_0$ and $z_1$ measured by the metric $\rho_D(z)|dz|$.

Let $\varphi$ be a non-vanishing analytic function on a hyperbolic domain $D$, namely, $\varphi : D \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is holomorphic. Then we set

$$V_D(\varphi) = \sup_{z \in D} \rho_D(z)^{-1} \left| \frac{\varphi'(z)}{\varphi(z)} \right|.$$  

Note also that $V_D(\varphi)$ can be thought of the Bloch semi-norm of the (possibly multi-valued) function $\log \varphi$. The quantity $V_D(\varphi)$ measures the rate of growth of $\varphi$ compared with the hyperbolic metric. More precisely, we have the following characterization.

**Proposition 1.** Let $\varphi$ be a non-vanishing analytic function on a hyperbolic domain $D$ and let $c$ be a positive constant. Then $V_D(\varphi) \leq c$ if and only if the double inequality

$$\exp \left( -c d_D(z_0, z_1) \right) \leq \frac{|\varphi(z_1)|}{|\varphi(z_0)|} \leq \exp \left( c d_D(z_0, z_1) \right)$$

holds for every pair of points $z_0, z_1$ in $D$.

We list properties of this quantity.

**Theorem 2.** Let $D$ be a hyperbolic domain and let $\varphi$ and $\psi$ be non-vanishing analytic functions on $D$.

(i) $V_D(\varphi \cdot \psi) \leq V_D(\varphi) + V_D(\psi)$.

(ii) $V_D(\varphi^n) = |\alpha|V_D(\varphi)$ holds for a complex number $\alpha$ whenever $\varphi^n$ is single-valued in $D$.

(iii) Let $p : D_0 \to D$ be an analytic (unbranched and unlimited) covering projection. Then $V_{D_0}(\varphi \circ p) = V_D(\varphi)$. In particular, $V_D(\varphi)$ is conformally invariant in the sense that this does not depend on the source domain.

(iv) $V_D(L \circ \varphi) = V_D(\varphi)$ holds for any conformal automorphism $L$ of $\mathbb{C}^*$. In particular, $V_D(1/\varphi) = V_D(\varphi) = V_D(c\varphi)$ for any constant $c \in \mathbb{C}^*$.

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(v) Let \( \omega : D_0 \to D \) be a holomorphic map. Then \( V_{D_0}(\varphi \circ \omega) \leq V_D(\varphi) \).

(vi) If \( \psi : D \to \mathbb{C}^* \) is univalent and if \( \varphi(D) \subset \psi(D) \) then \( V_D(\varphi) \leq V_D(\psi) \).

2. Circular width

Let \( \Omega \) be a hyperbolic plane domain with \( 0 \in \mathbb{C} \setminus \Omega \). The quantity

\[
W(\Omega) = \left( \inf_{w \in \Omega} |w| \rho_\Omega(w) \right)^{-1}
\]

will be called the circular width of \( \Omega \) (about the origin). This quantity can be expressed in terms of a covering projection onto \( \Omega \).

**Lemma 3.** Let \( \Omega \) be a proper subdomain of the punctured plane \( \mathbb{C}^* \) and let \( p \) be an analytic (unbranched) covering projection of a domain \( D \) onto \( \Omega \). Then \( W(\Omega) = V_D(p) \).

**Remark.** In general, we can define the circular width \( W_\alpha(\Omega) \) about a point \( \alpha \in \mathbb{C} \setminus \Omega \) in the same way: \( W_\alpha(\Omega) = 1/\inf_{w \in \Omega} |w - \alpha| \rho_\Omega(w) \). It is known that the domain constant \( C(\Omega) = \sup_{\alpha \in \mathbb{C} \setminus \Omega} W_\alpha(\Omega) \) is finite if and only if the set \( \mathbb{C} \setminus \Omega \) is uniformly perfect (see, for example, \([10]\) or \([12]\)). In this context, the constant \( W_\alpha(\Omega) \) appeared essentially in a paper \([13]\) by J.-H. Zheng.

We now collect basic properties of the circular width. Before that, we recall the notion of circular symmetrization. For a subdomain \( \Omega \) of \( \mathbb{C}^* \) we define the circular symmetrization \( \Omega^* \) (about the origin) by

\[
\Omega^* = \{ re^{i\theta} : \theta \in I(r, \Omega), 0 < r < \infty \},
\]

where \( I(r, \Omega) \) denotes the interval in the form \((- t/2, t/2)\) of the same length as \( I_r = \{ \theta \in [- \pi, \pi] : re^{i\theta} \in \Omega \} \) if \( I_r \neq [- \pi, \pi] \) otherwise \( I(r, \Omega) = [- \pi, \pi] \).

**Theorem 4.** Let \( \Omega \) and \( \Omega' \) be proper subdomains of the punctured plane \( \mathbb{C}^* \).

(i) \( W(\Omega) = W(L(\Omega)) \) for any conformal automorphism \( L \) of \( \mathbb{C}^* \).

(ii) If \( \Omega \subset \Omega' \), then \( W(\Omega) \leq W(\Omega') \).

(iii) Circular symmetrization does not decrease circular width: \( W(\Omega) \leq W(\Omega^*) \).

(iv) If \( \Omega \) is simply connected, then \( W(\Omega) \leq 4 \).

In general, the circular width may not be finite. We give here a characterization of domains with infinite circular width. In particular, if the origin is an isolated boundary point of \( \Omega \), then \( W(\Omega) = \infty \).

**Proposition 5.** Let \( \Omega \) be a proper subdomain of the punctured plane \( \mathbb{C}^* \). The circular width \( W(\Omega) \) is infinite if and only if there is a sequence of annuli \( A_n = \{ w \in \mathbb{C} : r_n < |w| < R_n \} \) with \( A_n \subset \Omega \) such that \( R_n/r_n \to \infty \).

The circular width may not behave continuously in \( \Omega \). For instance, consider the sequence of domains \( \Omega_n = \{ |w - 1| < 1 + 1/n \} \). Then \( \Omega_n \) converges to \( \Omega_\infty = \{ |w - 1| < 1 \} \) in the Hausdorff topology. But \( W(\Omega_n) = \infty \) by Proposition 5 whereas \( W(\Omega_\infty) \leq W(\mathbb{H}) = 2 \) (see Example 3.1). We can, however, show a continuity property of circular width in the following form.

**Proposition 6.** Let \( \Omega_n \) be a sequence of domains with \( \Omega_n \subset \Omega_{n+1} \) such that the union \( \Omega = \bigcup_{n=1}^\infty \Omega_n \) is a proper subdomain of \( \mathbb{C}^* \). Then \( W(\Omega_n) \to W(\Omega) \) as \( n \to \infty \).

The circular width \( W(\Omega) \) dominates the quantity \( V_D(\varphi) \) for holomorphic maps \( \varphi : D \to \Omega \).

**Theorem 7.** Let \( \varphi : D \to \Omega \) be holomorphic. Then \( V_D(\varphi) \leq W(\Omega) \).

**Proof.** By Theorem 2 (v), we have \( V_D(\varphi) = V_D(id_D \circ \varphi) \leq V_D(id_D) = W(\Omega) \).

Combining this with Proposition 1, we have the following. A similar result was obtained by \([13]\).

**Corollary 8.** Under the same hypotheses in Theorem 7,

\[
\exp \left( - W(\Omega) d_D(z_0, z_1) \right) \leq \frac{|\varphi(z_1)|}{|\varphi(z_0)|} \leq \exp \left( W(\Omega) d_D(z_0, z_1) \right), \quad z_0, z_1 \in D.
\]
3. Computations of circular widths

In the present section, we give exact values of circular width for several concrete examples. In view of Theorem 4 (iii), we see that circularly symmetric domains are particularly important.

**Example 3.1** (sectors). For $S(\beta) = \{w : |\arg w| < \pi \beta / 2\}$, $0 < \beta \leq 2$, we have $W(S(\beta)) = 2\beta$.

**Example 3.2** (half-sectors). Let $S(\beta, r) = \{w : |\arg w| < \pi \beta / 2, |w| < r\}$ and $S'(\beta, r) = \{w : |\arg w| < \pi \beta / 2, |w| > 1/r\}$ for $0 < \beta \leq 2$ and $0 < r < \infty$. Then $W(S(\beta, r)) = W(S'(\beta, r)) = 2\beta$.

**Example 3.3** (annuli). For the annulus $A(r, R) = \{w : r < |w| < R\}, 0 < r < R < \infty$, we have $W(A(r, R)) = (2/\pi) \log(R/r)$.

**Example 3.4** (disks). Let $D(a, r) = \{w : |w - a| < r\}$ for $0 < r \leq a$. Then

$$W(D(a, r)) = \frac{2r/a}{1 + \sqrt{1 - (r/a)^2}}.$$

**Example 3.5** (parallel strips). Let $P(a, b) = \{w : a < \Re w < b\}$ for $0 \leq a < b < \infty$. Then

$$W(P(a, b)) = \max_{0 \leq \theta \leq \pi/2} \frac{2t \cos \theta}{1 - t^2},$$

where $t$ is a number with $0 < t \leq 2/\pi$ determined by $\pi t/2 = (b - a)/(b + a)$.

**Example 3.6** (truncated wedges). Let $S(\beta, r, R) = \{w : |\arg w| < \pi \beta / 2, r < |w| < R\}, 0 < \beta \leq 2, 0 < r < R < \infty$. Then

$$W(\Omega) = \frac{\log(R/r)}{(1 + t)K(t)},$$

where

$$K(t) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - t^2 x^2)}}$$

is the complete elliptic integral of the first kind and $0 < t < 1$ is a number such that

$$\frac{K(\sqrt{1 - t^2})}{K(t)} = \frac{2\pi \beta}{\log(R/r)}.$$

Note that the quantity $\mu(t) = (\pi/2)K(\sqrt{1 - t^2})/K(t)$ is the modulus of the Grötzsch ring $\mathbb{D} \setminus [0, t]$ for $0 < t < 1$ and decreasing from $+\infty$ to 0 (see, for example, [1]). Therefore, we can always take such a $t$ satisfying the above relation.

We remark that the essentially same observations for the last example were made by Avhadiev and Aksent’ev [2] though they did not make systematic use of circular width.

4. Applications

In this section, we give a few applications of circular width. More concrete applications can be found in [9] and [11].

Let us introduce some notation. For a locally univalent function $f$ on $\mathbb{D}$, the quantity $T_f = f''/f'$ is called the pre-Schwarzian derivative of $f$ and measured by the norm

$$\|T_f\|_0 = \sup_{z \in \mathbb{D}} |1 - |z|^2| |T_f(z)|.$$

Note that this can be described by $\|T_f\|_0 = V_2(f')$. Let $\mathcal{A}$ denote the class of holomorphic functions $f$ on $\mathbb{D}$ normalized by $f(0) = 0, f'(0) = 1$.

**Theorem 9.** Let $\Omega$ be a proper subdomain of the punctured plane $\mathbb{C}^*$ with $W(\Omega) < 2$. If $f \in \mathcal{A}$ satisfies $f'(\mathbb{D}) \subset \Omega$, then $|f(z)| < M, z \in \Omega$. Here $M$ is a constant depending only on $W(\Omega)$.

The constant $M$ can be given by

$$M = \lambda \left[ \psi \left( \frac{1}{2} \right) - \psi \left( \frac{1}{M} \right) \right] - 1,$$

where $\lambda = W(\Omega)/2$ and $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function (see [7]).

It may be interesting to find a characterization of such subdomains $\Omega$ of $\mathbb{D}$ that $f'(\mathbb{D}) \subset \Omega$ implies boundedness of $f \in \mathcal{A}$. Note that the condition $f'(\mathbb{D}) \subset \mathbb{H}$ implies univalence of $f$ (Noshiro-Warschawski
theorem). Recently, Chuaqui and Gevzirtz [6] gave a characterization of such subdomains $\Omega$ of $\mathbb{H}$ that $f'(D) \subset \Omega$ implies quasiconformal extensibility of $f \in A$.

We next apply Theorem 7 to the problem of quasiconformal extensibility. Our result is based the following theorem due to J. Becker. See, for sharpness, Becker and Pommerenke [5].

Theorem 10 (Becker [4]). Let $f \in A$ be locally univalent. If $\|T_f\|_D \leq 1$, then $f$ is univalent. Furthermore, if $\|T_f\|_D \leq k$ for $k \in [0, 1)$, then $f$ has a $K$-quasiconformal extension to the whole plane, where $K = (1 + k)/(1 - k)$.

Making use of this result, we can show the following.

Theorem 11. Suppose that a proper subdomain $\Omega$ of the punctured plane $\mathbb{C}^*$ satisfies $W(\Omega) \leq k$ for some $k \leq 1$. If $f'(D) \subset \Omega$ for $f \in A$, then $f$ is univalent and, moreover, $f$ has a $K$-quasiconformal extension to the whole plane when $K = (1 + k)/(1 - k) < \infty$.


Proof. As we noted, the condition $f'(D) \subset \Omega$ implies that $\|T_f\|_D \leq W(\Omega) \leq k$. We now apply Theorem 10 to deduce the assertions. $\square$

Combining this with examples presented in the previous section, we obtain a series of corollaries. (Remember the fact that circular width is invariant under rotations.) The first corollary was noted by Avhadiev and Aksent’ev [3, pp. 33–34] at least when $\gamma = 0$.

Corollary 12. Let $0 < k \leq 1$ and $f \in A$. If $|\arg f'(z) - \gamma| < \pi k/4$ in $|z| < 1$ for some real constant $\gamma$, then $f$ is univalent and, moreover, it extends to a $K$-quasiconformal mapping of the whole plane when $K = (1 + k)/(1 - k) < \infty$.

Note also that the condition $|\arg f'(z)| < M$, $|z| < 1$, implies quasiconformal extensibility of $f$ when $M < \pi/2$ (see [6]).

Corollary 13. Let $k, r, R$ be positive numbers with $0 < \log(R/r) \leq \pi k/2$ and $f \in A$. If $r < |f'(z)| < M$ in $|z| < 1$, then $f$ is univalent and, moreover, it extends to a $K$-quasiconformal mapping of the whole plane when $K = (1 + k)/(1 - k) < \infty$.

We omit the other examples.

REFERENCES


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