Hyperbolic Metric and Hypergeometric Functions

Toshiyuki Sugawa

Department of Mathematics,
Graduate School of Science,
Hiroshima University,
Higashi-Hiroshima, 739-8526 Japan
sugawa@math.sci.hiroshima-u.ac.jp

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Hyperbolic Metric. The hyperbolic metric of a (hyperbolic) Riemann surface R is a complete conformal metric $\rho_R(z)|dz|$ on R satisfying the equation

$$\Delta \log \rho_R = 4\rho_R^2.$$

If we write $\rho_R(z) = e^{\varphi(z)/2}$, the equation is equivalent to the *Liouville equation*

$$\Delta\varphi = 8e^{\varphi}.$$

If $z = 0$ is a puncture of R, the density $\varphi(z)$ behaves as

$$\varphi(z) = 2\log\frac{1}{|z|} - 2\log\log\frac{1}{|z|} + O(1).$$

Example. If we write $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$, we have

$$\rho_{\mathbb{D}}(z) = \frac{1}{1-|z|^2}.$$

Branch Singularities. If $w = z^n$ for an integer $n \geq 2$, the equation

$$\rho(w)\left|\frac{dw}{dz}\right| = \rho_{\mathbb{D}}(z)$$

gives the solution

$$\begin{aligned}\rho(w) &= \frac{1}{n|w|^{1-1/n}(1-|w|^{2/n})} \\ &= \frac{1}{n}|w|^{\frac{1}{n}-1}(1+o(1)) \quad (w \to 0).\end{aligned}$$

Therefore, $\varphi = 2\log\rho$ has the asymptotic behaviour like

$$\varphi(w) = 2\left(1-\frac{1}{n}\right)\log\frac{1}{|w|} + O(1)$$

as $w \to 0$. Geometrically, the metric $\rho(w)$ is obtained by indentifying the both sides of the hyperbolic sector with opening $2\pi/n$. The same idea may apply to the case when n is not necessarily an integer.

Conical Singularities. A conformal metric $\rho(z)|dz|$ on a Riemann surfaces R is said to have *conical singularity* at $p \in R$ with order $\alpha > 0$ if $\varphi = 2\log\rho$ satisfies

$$\varphi(z) = 2(1-\alpha^{-1})\log\frac{1}{|z|} + O(1)$$

as $z \to 0$ where z is a local chart around p with $z(p) = 0$. For this notion, see, for example, [5] and the references therein.

A function $\alpha_R : R \to [0, +\infty)$ is called a *singularity data* if the set of "singularities"

$$\mathrm{Sing}(\alpha_R) = \{p \in R; \alpha_R(p) \neq 1\}$$

is a discrete subset of R.

In this article, we will allow a conformal metric $\rho = \rho(z)|dz|$ on R to have a (closed) discrete subset where the density may be $+\infty$ or 0. A conformal metric ρ on R (in the above sense) is called the hyperbolic metric of R with singularity data α_R if the induced distance is complete, if $\varphi = 2\log\rho$ satisfies the Liouville equation on $R \setminus \mathrm{Sing}(\alpha_R)$, and if

$$\varphi(z) = 2(1-\alpha_R(p)^{-1})\log\frac{1}{|z|} + O(1)$$

for each $p \in \mathrm{Sing}(\alpha_R)$, where z is a local chart around p with $z(p) = 0$.

The Gauss-Bonnet formula gives the next hyperbolicity condition for α_R if there is a hyperbolic metric on R with it as singularity data:

$$\sum_{p \in \mathrm{Sing}(\alpha_R)} (1-\alpha_R(p)^{-1}) > 2.$$

When the above inequality is satisfied, α_R is called *hyperbolic*.

E. Picard [3] proved the existence of the hyperbolic metric with conical singularities when R is the Riemann sphere. Much later, Heins [1] proved it for general Riemann surfaces.

Altough Poincaré's approach using the Fuchsian differential equations could give good information on the metric, it seems that no concrete forms of the hyperbolic metric with conical singularities are known.

Hypergeometric Functions. The Gauss' hypergeometric function is defined by

$$F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n,$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ denotes the shifted factorial, for $c \neq -1, -2, \ldots$.

The Case of $R = \widehat{\mathbb{C}}$ and $\mathrm{Sing}(\alpha_R) = \{0, 1, \infty\}$. Assume that $\alpha_{\widehat{\mathbb{C}}}(0) = \alpha, \alpha_{\widehat{\mathbb{C}}}(1) = \beta, \alpha_{\widehat{\mathbb{C}}}(\infty) = \gamma$ and that the hyperbolicity condition

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} < 1$$

holds. Set

$$\begin{aligned} a &= (1-\alpha-\beta-\gamma)/2, \\ b &= (1-\alpha+\beta-\gamma)/2, \\ c &= 1-\alpha. \end{aligned}$$

Then, the Schwarz triangle function

$$H(z) = \frac{z^{1-c} F(a-c+1, b-c+1, 2-c; z)}{F(a,b,c;z)}$$

maps the upper half-plane onto the domain bounded by two segments and a circular arc forming the angles $\pi\alpha, \pi\beta, \pi\gamma$ at the vertices (see [2, p. 315]). Note also that

$$H(1) = \frac{\Gamma(2-c)\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(1-a)\Gamma(1-b)}.$$

Choose an $r > 0$ so that the circular arc intersects perpendicularly the circle $|z| = r$. Then $H(z)/r$ maps the upper half-plane onto the hyperbolic triangle in $\mathbb{D}$ with angles $\pi\alpha, \pi\beta, \pi\gamma$. Thus, the hyperbolic metric with the given singularity data is given by the pull-back of $\rho_{\mathbb{D}} = |dz|/(1-|z|^2)$ by $H(z)/r$, namely,

$$\rho(z) = \frac{r|H'(z)|}{r^2 - |H(z)|^2}.$$

Special Case $\alpha = \beta = \gamma$. In this case,

$$\begin{aligned} r &= \frac{H(1)}{\sqrt{\cos \pi\alpha}} \\ &= \frac{1}{\sqrt{\pi \cos \pi\alpha}} \cdot \frac{\Gamma(1+\alpha)\Gamma((1-\alpha)/2)}{\Gamma(1-\alpha)\Gamma((1+\alpha)/2)\Gamma(1/2+\alpha)}. \end{aligned}$$

For the proof of the above formula, we use the second cosine theorem

$$\cosh C = \frac{\cos \pi\alpha \cos \pi\beta + \cos \pi\gamma}{\sin \pi\alpha \sin \pi\beta},$$

where C is the hyperbolic length of the circular arc in $\mathbb{D}$ (see, for instance, [4, Theorem 8.5]).

For a general case, one can compute the radius r possibly with much efforts.

References

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