ON THE BOTTOM OF THE SPECTRUM OF A RIEMANN SURFACE OF INFINITE TOPOLOGICAL TYPE

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Abstract. In this note, we shall present a sufficient condition for the positivity of the bottom of the spectrum of a Riemann surface. In particular, we shall show that an open Riemann surface of bounded geometry and of finite genus has positive bottom of the spectrum.

1. Introduction

Let R be a hyperbolic Riemann surface endowed with the Poincaré (or hyperbolic) metric $\rho_R = \rho_R(z)|dz|$ of constant negative curvature -1. Although some authors prefer to use $\rho_R/2$ of curvature -4 instead of ρ_R, we adopt here ρ_R of curvature -1 following the tradition in the spectral geometry. The Laplace-Beltrami operator $-\Delta$ with respect to the hyperbolic metric acts on the space $C_c^\infty(R)$ of smooth real-valued functions on R with compact support. This operator is known to uniquely extend to a positive unbounded self-adjoint operator on $L^2(R)$.

In this note, we shall consider the bottom $\lambda(R)$ of the L^2-spectrum of the hyperbolic Riemann surface. This quantity can be described by Rayleigh's quotient:

$$\lambda(R) = \inf_{\varphi \in C_c^\infty(R)} \frac{\iint_R |\nabla\varphi|^2 d\mathrm{vol}}{\iint_R \varphi^2 d\mathrm{vol}}.$$

The bottom of the spectrum $\lambda(R)$ is important in relation with the critical exponent of convergence $\delta(R)$ of R. This quantity is defined as the infimum of numbers $\delta > 0$ such that

$$\sum_{\gamma \in \Gamma} \exp(-\delta d_\Delta(0, \gamma(0))) < \infty,$$

where Γ is a Fuchsian group acting on the unit disk Δ uniformizing R, i.e. $R \cong \Delta/\Gamma$ and d_Δ denotes the hyperbolic distance in Δ. Note here that this definition does not depend on the particular choice of Γ. The critical exponent of convergence of R is known to be equal to the Hausdorff dimension of the conical limit set of the Fuchsian group Γ (cf. [5]).

The following is known as the theorem of Elstrodt-Patterson-Sullivan.

Theorem 1.1 ([7]).

$$\lambda(R) = \begin{cases} \frac{1}{4} & \text{if } 0 \le \delta(R) \le \frac{1}{2}, \\ \delta(R)(1-\delta(R)) & \text{if } \frac{1}{2} \le \delta(R) \le 1. \end{cases}$$

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In particular, $\lambda(R) > 0$ if and only if $\delta(R) < 1$.

In the case when the surface is of finite topological type, it is known that the bottom of the spectrum is 0 if and only if the surface is of finite conformal type, in other words, a compact Riemann surface with finitely many points removed. On the other hand, it seems that only a few results are known in the case of infinite topological type. Among them, for plane domains, Fernández and Rodríguez proved the following remarkable result.

Theorem 1.2 ([2] and [3]). *If R is a hyperbolic plane domain of bounded geometry, then $\lambda(R) > 0$. Moreover, for a separated sequence (a_n) of R, i.e.,*

$$\inf_{n \neq m} d_R(a_n, a_m) > 0,$$

the domain $R' := R \setminus \{a_n\}$ satisfies $\lambda(R') > 0$, too.

In the above theorem, d_R denotes the distance in R determined by the hyperbolic metric ρ_R, that is, $d_R(a, b)$ is the infimum of the hyperbolic lengths (measured by ρ_R) of arcs in R joining a with b. For non-empty subsets A and B of R, we also denote by $d_R(A, B)$ the hyperbolic distance of A and B in R. And a hyperbolic Riemann surface R is called *of bounded geometry* if the injectivity radius of R is (uniformly) away from 0, in other words, positive is the infimum $L(R)$ of the hyperbolic lengths of those curves which are homotopically nontrivial in R. For plane domains, several equivalent conditions for boundedness of geometry are known (for example, see [6]).

Actually, the authors of [3] proved the above theorem by showing the hyperbolic isoperimetric inequality. Now we introduce a variant of Cheeger's isoperimetric constant of R :

$$h(R) := \sup_{D \in \mathcal{D}_R} \frac{|D|}{|\partial D|},$$

where $\mathcal{D}_R$ denotes the set of relatively compact subdomains of R with piecewise smooth boundary, $|D| = |D|_R = \iint_D \rho_R(z)^2 dxdy$ and $|\partial D| = |\partial D|_R = \int_{\partial D} \rho_R(z)|dz|$. We say that R satisfies the *hyperbolic isoperimetric inequality* if $h(R) < \infty$. The following result says that the validity of the hyperbolic isoperimetric inequality implies the positivity of $\lambda(R)$.

Theorem 1.3 (Cheeger's inequality).

$$\frac{1}{4h(R)^2} \leq \lambda(R).$$

In fact, it is also shown that $\lambda(R)h(R) \leq C$ for an absolute constant $C < 3/2$ in [3], therefore $\lambda(R) > 0$ if and only if $h(R) < \infty$.

Our main aim in this note is to generalize Theorem 1.2 to the case of finite genus.

Theorem 1.4 (Main Theorem 1). *Let R be a non-compact hyperbolic Riemann surface of bounded geometry. Suppose that the genus g of R is finite. Then, the isoperimetric constant $h(R)$ satisfies*

$$h(R) \leq 1 + \frac{2\pi \min\{2g, 1\}}{L(R)}. \tag{1.1}$$

In particular, $\lambda(R) > 0$.

The Riemann surface R satisfying the above hypothesis is roughly isometric to a plane domain D with $L(D) > 0$ in the sense of Kanai, thus the theorem of Kanai [4] tells us that Theorem 1.2 implies also $\lambda(R) > 0$. Nevertheless our main theorem seems to have its own right in that our statement is quantitative. Of course, the latter part of Theorem 1.2 can also be generalized for finite genus case. In this note, we give more general result with an explicit estimate in the case of finite genus.

Theorem 1.5 (Main Theorem 2). *Let R be a non-compact hyperbolic Riemann surface of bounded geometry and of finite genus and $A_1, A_2, \cdots$ a (finite or infinite) sequence of compact subsets of R such that there exist a sequence $x_1, x_2, \cdots$ in R and constants σ, τ and H satisfying the following conditions.*

1. $0 < 2\sigma < \tau < L(R)/2$ *and* $1 \le H < \infty$,
2. $d_R(x_k, x_l) \ge \tau$ *if* $k \ne l$,
3. $A_n \subset \{x \in R; d_R(x, x_n) \le \tau - 2\sigma\}$, *and*
4. $h(B_n \setminus A_n) \le H$,

where $B_n = B(x_n, \tau) = \{x \in R; d_R(x, x_n) < \tau\}$. *Then* $R' = R \setminus \cup_{n=1}^{\infty} A_n$ *satisfies* $h(R') \le K < \infty$, *where K is a constant depending only on* $h(R), \sigma, \tau$ *and H. In particular* $\lambda(R') > 0$.

Remark 1. The constant K above is explicitly given in the proof in Section 3.

Remark 2. Note that B_n is simply connected because $\tau < L(R)$. Hence, by Corollary 2.2 in the following, we can see that the condition (4) is fulfilled if A_n is connected. Another sufficient condition for (4) can be given by application of Theorem 1.5 itself.

Taking closed disks $\bar{B}(x_n, \varepsilon_n)$ as A_n with $\varepsilon_n \to 0$, we have an example of the surface R' with $\lambda(R') > 0$ while $L^*(R') = L(R') = 0$, where $L^*(R')$ denotes the infimum of the hyperbolic lengths of closed geodesics in R'. Note that a similar exapmle was given in [2].

We should remark that the assumption of finiteness of genus cannot be eliminated in our main theorems. In fact, Brooks showed the following result.

Theorem 1.6 (Brooks [1]). *Let R be a compact Riemann surface of genus $g > 1$ and $F : \hat{R} \to R$ be a holomorphic unbranched Galois covering. Then $\lambda(\hat{R}) = 0$ if and only if the covering transformation group* $G = \{\gamma \in \mathrm{Aut}(\hat{R}); F \circ \gamma = F\} \cong \pi_1(R, *)/\pi_1(\hat{R}, *)$ *is amenable.*

For the definition of amenability, see [1], for example. Here we only cite the fact that if G is abelian then amenable while G contains a free group with two generators then non-amenable.

We also remark that $L(\hat{R}) \ge L(R) > 0$ in the above case, so the boundedness of geometry need not imply the positivity of the bottom of the spectrum in the case of infinite genus.

It is easy to show that if R is of finite conformal type and $F : \hat{R} \to R$ is a holomorphic unbranched Galois covering with amenable covering transformation group then $\lambda(\hat{R}) = 0$. In particular, since $\hat{R} = \mathbb{C} \setminus \mathbb{Z} \to (\mathbb{C} \setminus \mathbb{Z})/\mathbb{Z} = \mathbb{C} \setminus \{0, 1\}$ is a $\mathbb{Z}$-cover, thus amenable cover, we have $\lambda(\hat{R}) = 0$. On the other hand, evidently $L^*(\hat{R}) > 0$. Therefore the condition $L^*(R) > 0$ need not guarantee the positivity of $\lambda(R)$.

This article is organized as follows. In Section 2, we will show a fundamental estimate of the hyperbolic area of a relatively compact subdomain by the method of M. Suzuki [8], from which Theorem 1.4 follows.

Section 3 is devoted to the proof of Theorem 1.5. Essential idea in the proof is the same as in [3], but we need more efforts.

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2. Estimate of hyperbolic area

Basically following the method in [8] by M. Suzuki as is indicated in [2], we shall make an estimate of the hyperbolic area of a relatively compact subdomain of a hyperbolic Riemann surface by the length of its boundary.

Let R be a hyperbolic Riemann surface with the hyperbolic metric $\rho_R = \rho_R(z)|dz|$ of constant negative curvature -1, i.e., there exists a holomorphic universal covering map $f : \varDelta \to R$ from the unit disk $\varDelta$ onto R and ρ_R is the Riemannian metric determined by $\frac{2|dz|}{1-|z|^2} = f^*\rho_R$. Note that the hyperbolic metric ρ_R is independent of the particular choice of the universal covering map f. We denote by $\varGamma$ the deck transformation group $\{\gamma \in \mathrm{Aut}(\varDelta) = \mathrm{PSU}(1,1); f \circ \gamma = f\}$, thus $\varGamma$ is the Fuchsian group which uniformizes R.

Now we show the following

Lemma 2.1. *Let D be a relatively compact subdomain with piecewise smooth boundary in an arbitrary hyperbolic Riemann surface R. Then,*

$$|D| \leq |\partial D| + 2\pi(m + 2k - 1). \tag{2.1}$$

where m is the number of boundary components of D and k is the genus of D.

Proof. If D has trivial (=contractible) boundary curves then these curves bound disks in R, so we obtain a new domain by atatching D the disks bounded by these curves. In this procedure, the hyperbolic length of the boundary and the number of boundary components decrease while the hyperbolic area increases. Hence, in order to prove (2.1), we may assume that D has no trivial boundary curves.

Let $a_1, \cdots, a_m$ be the boundary curves of D. Fix a point x_0 in a_m, then it is easy to show that there exist simple smooth arcs $b_1, \cdots, b_{2k+m-1}$ such that b_j starts from and ends at x_0 for $j = 1, \cdots, 2k$ and starts from x_0 and ends at a point in a_{j-2k} for $j = 2k+1, \cdots, 2k+m-1$, and each arc is contained in R and does not intersect other arcs except for its end points. Therefore, $D' = D \setminus \cup_{j=1}^{2k+m-1} b_j$ is a relatively compact simply connected subdomain of R. Let $\hat{D}'$ be a connected component of $f^{-1}(D')$. Then $f : \hat{D}' \to D'$ is biholomorphic and $\hat{D}'$ is a Jordan domain bounded by the union of simple closed arcs $\hat{a}_1, \cdots, \hat{a}_m$ and $\hat{b}_1^+, \hat{b}_1^-, \cdots, \hat{b}_{2k+m-1}^+, \hat{b}_{2k+m-1}^-$, where $\hat{a}_j$ is a lift of a_j and $\hat{b}_j^+$ and $\hat{b}_j^-$ are lifts of b_j. We note here that there exists a $\gamma_j \in \varGamma \setminus \{1\}$ which maps $\hat{b}_j^+$ onto $\hat{b}_j^-$ with reversing orientation for each $j = 1, \cdots, 2k+m-1$.

Since $f : \varDelta \to R$ is a local isometry with respect to the hyperbolic metric, we have $|D|_R = |D'|_R = |\hat{D}'|_\varDelta$ and $|a_j|_R = |\hat{a}_j|_\varDelta$. Noting that the hyperbolic density function $\rho_0(z) = \frac{2}{1-|z|^2}$ of $\varDelta$ satisfies

$$\rho_0^2 dx \wedge dy = -\Delta \log \rho_0 dx \wedge dy = -d^* d \log \rho_0,$$

where ${}^*dF = -\frac{\partial F}{\partial y} dx + \frac{\partial F}{\partial x} dy$, we can see by Stokes' theorem that

$$|D|_R = \iint_{\hat{D}'} \frac{4dx \wedge dy}{(1-|z|^2)^2} = -\iint_{\hat{D}'} d^* d \log \rho_0 = -\int_{\partial \hat{D}'} {}^*d \log \rho_0.$$

Since

$${}^*d \log \rho_0 = -i \frac{\bar{z}dz - zd\bar{z}}{1-|z|^2} = \mathrm{Im} \left(\frac{2\bar{z}dz}{1-|z|^2} \right) = -\mathrm{Im}\omega,$$

where $\omega = -2\bar{z}(1-|z|^2)^{-1}dz$, we have

$$|D|_R = \mathrm{Im} \int_{\partial \hat{D}'} \omega = \sum_{j=1}^{m} I_j + \sum_{l=1}^{2k+m-1} J_l.$$

In the above, $I_j = \mathrm{Im} \int_{\hat{a}_j} \omega$ and $J_l = \mathrm{Im}(\int_{\hat{b}_l^+} \omega + \int_{\hat{b}_l^-} \omega)$.

First, we note that $|I_j| \le \int_{\hat{a}_j} \frac{2|dz|}{1-|z|^2} = |a_j|_R$. Next, since we can write γ_l as $\gamma_l(z) = \frac{\bar{\alpha}z+\bar{\beta}}{\beta z+\alpha}$ for constants α and β with $|\alpha|^2 - |\beta|^2 = 1$, we see that

$$\omega - \gamma_l^* \omega = \frac{2\beta dz}{\beta z + \alpha} = 2d \log(\beta z + \alpha).$$

We then get that

$$\begin{aligned} J_l &= \mathrm{Im} \left(\int_{\hat{b}_l^+} \omega - \int_{\gamma_l(\hat{b}_l^+)} \omega \right) = \mathrm{Im} \int_{\hat{b}_l^+} (\omega - \gamma_l^* \omega) \\ &= 2\mathrm{Im} \int_{\hat{b}_l^+} d \log(\beta z + \alpha) = 2 \int_{\hat{b}_l^+} d \arg(\beta z + \alpha). \end{aligned}$$

Because the disk $\{\beta z + \alpha; |z| < 1\}$ does not contain the origin, we have $|J_l| < 2\pi$. From these observations, we can conclude that

$$|D|_R \le \sum_{j=1}^{m} |a_j|_R + 2\pi(2k+m-1) = |\partial D|_R + 2\pi(2k+m-1).$$

Now the proof is completed. $\square$

By this lemma, we also have the following result, which will be used later. This statement can be found in [3] but the proof is omitted there, so we include it for convenience of the reader.

Corollary 2.2 (cf. [3]). *For a simply or doubly connected hyperbolic Riemann surface R it follows that $h(R) = 1$.*

Proof. When R is simply connected, we may assume that R is the unit disk Δ. By the above lemma, $|D| \leq |\partial D|$ for any $D \in \mathcal{D}_R$, hence $h(R) \leq 1$. On the other hand, for $D_r = \{z \in \Delta; |z| < r\}$, we can calculate that $|D_r|_\Delta = 4\pi r^2(1-r^2)^{-1}$ and $|\partial D_r|_\Delta = 4\pi r(1-r^2)^{-1}$, so we see that $h(R) = 1$.

When R is doubly connected, the result does not follow directly from the above lemma. To see $h(R) \leq 1$, it suffices to show that $|D| \leq |\partial D|$ only for doubly connected domains D in $\mathcal{D}_R$ without trivial boundary components. Fix a smooth arc b in D connecting both of boundary components of D. Then $D' = D \setminus b$ is simply connected. Let $f : \Delta \to R$ be a holomorphic universal covering map and Γ its covering transformation group. Then Γ is generated by a single element, say γ. Let W be a connected component of $f^{-1}(D')$. Then ∂W consists of a lift $\hat{a}_1, \hat{a}_2$ of boundary curves a_1, a_2 of D and lifts $\hat{b}^+$ and $\hat{b}^-$ of b, where $\gamma(\hat{b}^-) = \hat{b}^+$. We denote by W_n the interior of $\cup_{j=0}^{n-1} \gamma^j(\overline{W})$ for $n = 1, 2, \cdots$. Note here that $\partial W_n = \cup_{j=0}^{n-1} \gamma^j(\hat{a}_1 \cup \hat{a}_2) \cup \gamma^n(\hat{b}^-) \cup \hat{b}^-$. Since W_n is simply connected, the above lemma yields that

$$n|D|_R = |W_n|_\Delta \leq |\partial W_n|_\Delta = n|\hat{a}_1|_\Delta + n|\hat{a}_2|_\Delta + 2|\hat{b}^-|_\Delta = n|\partial D|_R + 2|b|_R.$$

Letting $n \to \infty$, we then get $|D| \leq |\partial D|$, thus $h(R) \leq 1$. Actually, one can show that $h(R) = 1$, as above. For example, if R is of finite modulus, we may assume that $R = \{r < |z| < 1/r\}$ with $0 < r < 1$. Then $D_s = \{s < |z| < 1/s\}$ with $s = r^{2\theta/\pi}$ satisfies that $|D_s| = 2\ell \tan\theta$ and $|\partial D_s| = 2\ell/\cos\theta$, where $\ell = \pi^2/\log 1/r$, thus $|D_s|/|\partial D_s| = \sin\theta$. This shows that $h(R) \geq 1$. □

Proof. Proof of Theorem 1.4 It is enough to show the hyperbolic isoperimetric inequality only for $D \in \mathcal{D}_R$ without trivial boundary components. Then, recalling that $L(R)$ is the infimum of hyperbolic lengths of non-trivial loops in R, we have $|\partial D| \geq mL(R) \geq L(R)$, where m denotes the number of boundary components of D. Let k be the genus of D. Then $k \leq g$, thus by Lemma 2.1 it holds that

$$|D| \leq |\partial D| + 2\pi(m + 2k - 1) \leq \left(1 + \frac{2\pi}{L(R)}\right)|\partial D| + 2\pi(2g-1).$$

In the case of $g = 0$, we immediately obtain (1.1). When $g \geq 1$, it follows that

$$|D| \leq \left(1 + \frac{2\pi}{L(R)}\right)|\partial D| + 2\pi(2g-1)\frac{|\partial D|}{L(R)} = \left(1 + \frac{4\pi g}{L(R)}\right)|\partial D|,$$

thus now (1.1) is proved. □

Remark . As is seen from the above proof, we have slightly more general result as follows. *Let R be a conformally infinite Riemann surface, i.e., of infinite hyperbolic area. Suppose that the genus g and the number n of punctures of R are finite and the infimum $L^*(R)$ of the hyperbolic lengths of closed geodesics in R is positive. Then we have*

$$h(R) \leq 1 + \frac{2\pi \min\{2g+n, 1\}}{L^*(R)}.$$

3. Estimate for subdomains with totally geodesic boundary

In this section, we explain another approach for estimation of the hyperbolic area of subdomains by its boundary length. Now we introduce another class of subdomains, which is canonical in some sense, and easier to treat than $\mathcal{D}_R$.

We suppose that the hyperbolic Riemann surface R is not simply nor doubly connected, i.e., the fundamental group of R is not abelian. (Otherwise, we know already that $h(R) = 1$, so have nothing to do.) Let $\mathcal{D}_R^{\text{geod}}$ be the set of those subdomains of R whose end consists of finitely many pairwise disjoint simple closed geodesics in R and finitely many punctures. In other words, $D \in \mathcal{D}_R^{\text{geod}}$ if and only if D is of finite topological type (k, n, m), where k is the genus and n and m the numbers of punctures and holes, respectively, of D and the relative boundary ∂D of D in R consists of m simple closed geodesics of R. We remark that the above D is not neccesarily relatively compact in R but has finite hyperbolic area $2\pi(2k+m+n-2)$. In fact, the double $\widetilde{D}$ of D is of finite conformal type $(G, N) = (2g+m-1, 2n)$, so has hyperbolic area $2\pi(2G+N-2) = 4\pi(2g+m+n-2)$, thus $|D|_R = |\widetilde{D}|/2 = 2\pi(2g+m+n-2)$. Now we define the auxiliary constant $h^{\text{geod}}(R)$ by

$$h^{\text{geod}}(R) = \sup_{D \in \mathcal{D}_R^{\text{geod}}} \frac{|D|_R}{|\partial D|_R}.$$

The following is essentially due to Fernández-Rodríguez [3] and will be the key to our argument here.

Lemma 3.1. *For a hyperbolic Riemann surface R with non-abelian fundamental group,*

$$h^{\text{geod}}(R) \le h(R) \le h^{\text{geod}}(R) + 2.$$

Proof. The left-hand side inequality immediately follows from the definition. We now show the right-hand side. Let D be in $\mathcal{D}_R$. In order to show $|D| \le (h^{\text{geod}}(R)+2)|\partial D|$, we may assume that D has no trivial boundary components. Then each boundary component a_j of D is freely homotopic to either a closed geodesic b_j or a puncture P_j. We denote by D_1 the domain obtained from D by replacing its boundary components (or ends) a_j by b_j or P_j. By assumption, D_1 is non-degenerate, so $D_1 \in \mathcal{D}_R^{\text{geod}}$. Clearly $|\partial D_1| \le |\partial D|$ and the difference $D_1 \setminus D$ consists of simply or doubly connected components W_j's, thus Lemma 2.2 implies

$$|D_1| - |D| \le |D_1| - |D \cap D_1| = \sum_j |W_j| \le \sum_j |\partial W_j| \le |\partial D_1| + |\partial D| \le 2|\partial D|,$$

which proves the lemma. □

Remark . Lemma 3.1 also proves Theorem 1.4 with slightly different estimate:

$$h(R) \le 2 + \frac{2\pi \max\{2g-1, 1\}}{L(R)}.$$

One may see that this estimate is nearly optimal.

We need also the following elementary fact.

Lemma 3.2. *Let R be a hyperbolic Riemann surface and S its subdomain. For a subset X of S we set $\delta = d_R(X, \partial S) = \inf\{d_R(x,s); x \in X, s \in \partial S\}$. Then it holds that $1 \leq \rho_S/\rho_R \leq \coth\delta$ on X.*

Proof. Take any point $x_0 \in X$ and fix it. Let $f : \Delta \to R$ be a holomorphic universal covering map of R with $f(0) = x_0$. Then, by assumption, $\Delta_r := \{|z| < r\} = \{z \in \Delta; d_\Delta(0,z) < \delta\}$ is contained in $\hat{S} := f^{-1}(S)$, where $r = \tanh\delta$. Then the Schwarz-Pick lemma implies that

$$1 \leq \rho_S(x_0)/\rho_R(x_0) = \rho_{\hat{S}}(0)/\rho_\Delta(0) = \rho_{\hat{S}}(0) \leq \rho_{\Delta_r}(0) = 1/r = \coth\delta,$$

thus the proof is now finished. □

Proof. Proof of Theorem 1.5 In the following, let R and R' be as in Theorem 1.5 as well as $A_n, x_n, \sigma, \tau, H$ and B_n. We fix $D \in \mathcal{D}^{\mathrm{geod}}_{R'}$.

Set $B'_n = \{x \in R; d_R(x, x_n) < \tau - \sigma\}$ and $N = \{n; D \cap B'_n \neq \emptyset\}$. Then $|D|_{R'} = |D \cap R''|_{R'} + \sum_{n\in N} |D \cap B'_n|_{R'}$, where $R'' = R \setminus \cup_{n=1}^{\infty} \bar{B}'_n$. Note that $d_R(R'', \partial R') \geq \sigma$ and that $d_{R'}(B'_n \setminus A_n, \partial B_n) \geq d_R(B'_n \setminus A_n, \partial B_n) \geq \sigma$. By Lemma 3.2, we can estimate as

$$\begin{aligned}
|D \cap R''|_{R'} &\leq \coth^2\sigma \cdot |D \cap R''|_R \leq h(R)\coth^2\sigma \cdot |\partial(D \cap R'')|_R \\
&\leq h(R)\coth^2\sigma \cdot |\partial(D\cap R'')|_{R'} \\
&= h(R)\coth^2\sigma(|\partial D \cap R''|_{R'} + \sum_{n\in N} |D \cap \partial B'_n|_{R'})
\end{aligned}$$

and

$$\begin{aligned}
|D \cap B'_n|_{R'} &\leq |D \cap B'_n|_{B_n\setminus A_n} \leq h(B_n \setminus A_n)|\partial(D \cap B'_n)|_{B_n\setminus A_n} \\
&\leq H\coth\sigma(|\partial D \cap B'_n|_{R'} + |D \cap \partial B'_n|_{R'}).
\end{aligned}$$

Here we can further see that for each $n \in N$,

$$\begin{aligned}
|D \cap \partial B'_n|_{R'} &\leq |\partial B'_n|_{R'} \leq \coth\sigma|\partial B'_n|_R = \coth\sigma \cdot 2\pi\sinh(2\tau - 2\sigma) \\
&\leq \pi\sigma^{-1}\coth\sigma\sinh(2\tau-2\sigma)|\partial D \cap B_n|_{R'},
\end{aligned}$$

because $|\partial D \cap B_n|_{R'} \geq 2d_{R'}(B'_n, \partial B_n) \geq 2\sigma$. By summing up these estimates, we obtain

$$\begin{aligned}
|D|_{R'} &\leq h(R)\coth^2\sigma|\partial D \cap R''|_{R'} + H\coth\sigma\sum_{n\in N}|\partial D \cap B'_n|_{R'} \\
&\quad + (h(R)\coth^2\sigma + H\coth\sigma)\cdot\pi\sigma^{-1}\coth\sigma\sinh(2\tau-2\sigma)|\partial D \cap B_n|_{R'} \\
&\leq (h(R)\coth^2\sigma + H\coth\sigma)(1 + \pi\sigma^{-1}\coth\sigma\sinh(2\tau-2\sigma))|\partial D|_{R'}.
\end{aligned}$$

The last inequality shows that

$$\begin{aligned}
h(R') &\leq h^{\mathrm{geod}}(R') + 2 \\
&\leq (h(R)\coth^2\sigma + H\coth\sigma)(1 + \pi\sigma^{-1}\coth\sigma\sinh(2\tau-2\sigma)) + 2.
\end{aligned}$$

□

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