A REMARK ON AHLFORS' UNIVALENCE CRITERION

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Abstract. In this note, we will remove an additional assumption made for Ahlfors' univalence criterion. This leads to an estimate of the inner radius of univalence for an arbitrary quasidisk in terms of a quasiconformal reflection.

1. Introduction

Let D be a domain in the Riemann sphere $\widehat{\mathbb{C}}$ with the hyperbolic metric $\rho_D(z)|dz|$ of constant negative curvature -4. For a holomorphic function φ on D, we define the hyperbolic sup-norm of φ by

$$\|\varphi\|_D = \sup_{z\in D} \rho_D(z)^{-2}|\varphi(z)|.$$

We denote by $B_2(D)$ the complex Banach space consisting of all holomorphic functions of finite hyperbolic sup-norm. For a holomorphic map $g : D_1 \to D_2$, the pullback $g^* : \varphi \mapsto \varphi \circ g \cdot (g')^2$ is a linear contraction from $B_2(D_2)$ to $B_2(D_1)$. In particular, if g is biholomorphic, $g^* : B_2(D_2) \to B_2(D_1)$ becomes an isometric isomorphism. As is well known, the Schwarzian derivative $S_f = (f''/f')' - (f''/f')^2/2$ of a univalent function on D satisfies $\|S_f\|_D \le 12$ (see [?]). This result is classical for the unit disk $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$, actually, a better estimate $\|S_f\|_{\mathbb{D}} \le 6$ holds. On the other hand, Nehari's theorem [?] asserts that if a locally univalent function f on $\mathbb{D}$ satisfies $\|S_f\|_{\mathbb{D}} \le 2$, then f is necessarily univalent. Hille's example [?] shows that the number 2 is best possible. We now define the quantity $\sigma(D)$, which is called the *inner radius of univalence* of D, as the infimum of the norm $\|S_f\|_D$ of those locally univalent meromorphic function f on D which are not globally univalent. In other words, $\sigma(D)$ is the possible largest value $\sigma \ge 0$ with the property that the condition $\|S_f\|_D \le \sigma$ implies univalence of f in D. In the case $D = \mathbb{D}$, we already know $\sigma(\mathbb{D}) = 2$. For a comprehensive exposition of these notions and some background, we refer the reader to the book [?] of O. Lehto.

Ahlfors [?] showed that every quasidisk has positive inner radius of univalence. Conversely, Gehring [?] proved that if a simply connected domain has positive inner radius of univalence then it must be a quasidisk. Later, Lehto [?] pointed out the inner radius of univalence of a quasidisk can be estimated by the Ahlfors method as

$$\sigma(D) \ge 2 \inf_{z\in D'} \frac{|\bar\partial\lambda(z)| - |\partial\lambda(z)|}{|\lambda(z) - z|^2 \rho_D(z)^2}, \tag{1}$$

where λ is a quasiconformal reflection in ∂D which is continuously differentiable off ∂D and $D' = D \setminus \{\infty, \lambda(\infty)\}$. However, in order to obtain the estimate (1) rigorously, a kind of approximation procedure must work, so an additional assumption was needed. For

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example, Lehto [?, Lemma III.5.1] assumed the quasidisk D to be exhausted by domains of the form $\{rz; z \in D\}$ for $0 < r < 1$. More recently, Betker [?] gave a similar result for general quasidisks under the assumption that the quasiconformal reflection λ is of a special form associated with the Löwner chains. For another additional condition, see a remark at the end of the next section.

Remark that if we content ourselves with an estimate of the form $\sigma(D) \geq C(K)$ for a K-quasidisk D, where $C(K)$ is a positive constant depending only on K, the original idea of Ahlfors [?] is sufficient (see also [?, Chapter VI] and [?, Theorem II.4.1]).

Our main result is to show (1) without any additional assumption, which might be known as a kind of folklore.

Theorem 1. *Let D be a quasidisk with a quasiconformal reflection λ in ∂D which is continuously differentiable off ∂D. Then the inequality* (1) *holds for D.*

2. Proof of main result

First of all, we make a quick review of the original proof of (1) by Ahlfors under an additional assumption. Let a quasiconformal reflection λ in ∂D be given, i.e., λ is an orientation-reversing homeomorphic involution of $\widehat{\mathbb{C}}$ keeping each boundary point of D fixed and satisfying that $\lambda(\bar{z})$ is quasiconformal. Further suppose that λ is continuously differentiable on $\widehat{\mathbb{C}} \setminus \partial D$. We note that $|\partial\lambda| \leq k_0 |\bar{\partial}\lambda|$ for some constant $0 \leq k_0 < 1$.

Noting that the inequality (1) is invariant under a Möbius transformation (see [?, Sec. II 4.1]), we may assume that a quasidisk D is contained in $\mathbb{C}$. We take a $\varphi \in B_2(D)$ with $\|\varphi\|_D < \varepsilon_0$, where ε_0 denotes the right-hand side of the inequality (1).

Let η_0 and η_1 be the solutions of the linear differential equation

$$2y'' + \varphi y = 0 \tag{2}$$

in D with the initial conditions $\eta_0 = 1,\ \eta_0' = 0$ and $\eta_1 = 0,\ \eta_1' = 1$, respectively, at a reference point w_0 in D. Then it is well known that $f = \eta_1/\eta_0$ satisfies the Schwarzian differential equation $S_f = \varphi$ in D and has the normalization $f(w_0) = f'(w_0) - 1 = f''(w_0) = 0$. To extend f to the whole sphere, we consider the map

$$F(w) = \frac{\eta_1(w) + (\lambda(w) - w)\eta_1'(w)}{\eta_0(w) + (\lambda(w) - w)\eta_0'(w)}.$$

A direct cumputation shows that F is a local C^1 diffeomorphism in D and satisfies

$$\frac{\partial F(w)}{\bar{\partial} F(w)} = \frac{\partial\lambda(w) + (\lambda(w) - w)^2 S_f(w)/2}{\bar{\partial}\lambda(w)}.$$

Hence the assumption $\|\varphi\|_D < \varepsilon_0$ implies

$$\|\partial F/\bar{\partial} F\|_\infty \leq k := 1 - (1 - k_0)(1 - k_1) < 1,$$

where $k_1 = \|\varphi\|_D/\varepsilon_0 < 1$. Therefore, if the map

$$\hat{f}(w) = \begin{cases} f(w) & \text{for } w \in D, \\ F(\lambda(w)) & \text{for } w \in D^* = \widehat{\mathbb{C}} \setminus \overline{D} \end{cases} \tag{3}$$

continuously extends to the boundary ∂D and locally homeomorphic nearby there, $\hat{f}$ would become a local homeomorphism of the Riemann sphere, and hence a (quasiconformal) homeomorpshism of it.

In particular, if ∂D is an analytic Jordan curve and if the quasiconformal reflection λ is continuously differentiable at any point, it is easily verified that $\hat{f}$ is a local C^1-diffeomorphism of $\widehat{\mathbb{C}}$ and then a quasiconformal extension of f for φ holomorphic in $\overline{D}$, namely, holomorphic in a neighborhood of $\overline{D}$.

For a general $\varphi \in B_2(D)$, a crux will be the following, which is essentially due to Bers [?, Lemma 1].

Proposition 2. *Let D be a Jordan domain in $\widehat{\mathbb{C}}$. For any $\varphi \in B_2(D)$ there exists a sequence $(\varphi_j)_j$ of holomorphic functions in $\overline{D}$ such that $\|\varphi_j\|_D \le \|\varphi\|_D$ and φ_j tends to φ uniformly on each compact subset of D as $j \to \infty$.*

Proof. We denote by $g : \mathbb{D} \to D$ the Riemann mapping function of D with $g(0) = w_0$ and $g'(0) > 0$. Let D_j, $j = 1, 2, \dots$ be Jordan domains with $\overline{D}_{j+1} \subset D_j$ and with $\bigcap_j D_j = D$. Then the Carathéodory kernel theorem implies that the Riemann mapping functions g_j of D_j with $g_j(0) = w_0$ and $g_j'(0) > 0$ converge to g uniformly on each compact subset of the unit disk as j tends to ∞. Now we set $\varphi_j = (g \circ g_j^{-1})^* \varphi$. We then have $\|\varphi_j\|_D \le \|\varphi_j\|_{D_j} = \|\varphi\|_D$ by the Schwarz-Pick lemma: $\rho_D \ge \rho_{D_j}$. We also have $\varphi_j \to \varphi$ locally uniformly as $j \to \infty$. $\square$

With this result, the following lemma implies our main result.

Lemma 3. *Suppose that $\varphi \in B_2(D)$ with $\|\varphi\|_D \le k_1 \varepsilon_0$ is holomorphic in $\overline{D}$, where $0 \le k_1 < 1$ and ε_0 denotes the quantity in the right-hand side of (1). Then the function $\hat{f}$ defined by (3) extends to a K-quasiconformal homeomorphism of the Riemann sphere, where $K = (1+k)/(1-k)$ and $k = 1 - (1-k_0)(1-k_1)$.*

Actually, we can prove our main theorem as follows. Let $\varphi \in B_2(D)$ satisfy $\|\varphi\|_D < \varepsilon_0$ and set $k_1 = \|\varphi\|_D / \varepsilon_0$. We take a sequence $(\varphi_j)_j$ as in Proposition 2. Let $\hat{f}$ and $\hat{f}_j$ be the functions in $\widehat{\mathbb{C}} \setminus \partial D$ defined by (3) for φ and φ_j, respectively. Then, by the above lemma, each $\hat{f}_j$ can be continued to a K-quasiconformal homeomorphism of $\widehat{\mathbb{C}}$, where $K = (1+k)/(1-k)$ and $k = 1 - (1-k_0)(1-k_1)$. Since normalized K-quasiconformal mappings form a normal family, $\hat{f}_j$ has a subsequence converging to a K-quasiconformal mapping in $\widehat{\mathbb{C}}$ uniformly. By construction, the limit mapping coincides with $\hat{f}$ in $\widehat{\mathbb{C}} \setminus \partial D$. This implies that $\hat{f}$ has a K-quasiconformal extension to the whole sphere. Now the proof of our main theorem is complete except for a proof of the above lemma.

Remark. Under the assumption that φ is holomorphic in $\overline{D}$ with $\|\varphi\|_D < \varepsilon_0$, a direct calculation shows

$$\partial \hat{f}(z) = -\frac{1 + (z - \lambda(z))^2 \varphi(\lambda(z)) \partial \lambda(z)/2}{(\eta_0(\lambda(z)) + (z - \lambda(z)) \eta_0'(\lambda(z))^2} \quad \text{and}$$

$$\bar{\partial} \hat{f}(z) = -\frac{(z - \lambda(z))^2 \varphi(\lambda(z)) \bar{\partial} \lambda(z)/2}{(\eta_0(\lambda(z)) + (z - \lambda(z)) \eta_0'(\lambda(z))^2}$$

at every $z \in D^* \setminus \{\infty, \lambda(\infty)\}$. Therefore, if $(\lambda(z)-z)^2\bar\partial\lambda(z)$ vanishes at the boundary ∂D, then we would obtain continuous extensions of $\partial\hat f$ and $\bar\partial\hat f$ to $\widehat{\mathbb{C}}$. Moreover, the limits of

$$\frac{\hat f(z+t)-\hat f(z)}{t} \quad \text{and} \quad \frac{\hat f(z+it)-\hat f(z)}{t}$$

when t tending to 0 along the real axis both exist and are equal to $f'(z)$ and $if'(z)$, respectively for each $z \in \partial D$ because when $z+t$ or $z+it$ approaches to z in D^* the above quotients tend to the desired values by (4) below. This implies that our $\hat f$ has continuous partial derivatives everywhere in $\widehat{\mathbb{C}}$. Hence, in this case, we can conclude that $\hat f$ is a local C^1-diffeomorphism of $\widehat{\mathbb{C}}$, and then, a global C^1-diffeomorphism of it.

We note that it is always possible to take such a quasiconformal reflection λ for any quasidisk D (see [?] or [?, Section II.4]).

3. Proof of Lemma 3

Let φ be as in Lemma 3. Then the solutions η_0 and η_1 of (2) are holomorphic in $\overline{D}$. Thus $\hat f$ can be continuously extended to the whole sphere and $\hat f(\partial D)$ is locally a conformal image of a quasi-circle. Now we require an extension theorem for quasiregular mappings, where a continuous map f from a plane domain Ω into $\widehat{\mathbb{C}}$ is called (K-)quasiregular if f can be decomposed into the form $g \circ \omega$ where ω is a (K-)quasiconformal mapping on Ω and g is a non-constant meromorphic function on $\omega(\Omega)$ (see [?, Chapter VI] where the authors used the term "quasiconformal function" instead of "quasiregular mapping"). Note that a non-constant continuous function $f : \Omega \to \mathbb{C}$ is K-quasiregular if and only if f is ACL=ACL^1 and $|\bar\partial f| \le k|\partial f|$ a.e. in Ω, where $k = (K-1)/(K+1)$ (see [?]).

Lemma 4. *Let Ω be a plane domain and C an open quasi-arc (or a quasi-circle) in Ω such that $\Omega \setminus C$ is an open set in $\widehat{\mathbb{C}}$. Suppose that $f : \Omega \to \widehat{\mathbb{C}}$ is a continuous map such that $f|_{\Omega\setminus C}$ is a locally injective K-quasiregular map and that, for each $x \in C$, f maps $C \cap U$ injectively onto a quasi-arc for some open neighborhood U of x in Ω. Then f is K-quasiregular in Ω.*

Proof. If once we know that f is quasiregular in Ω, we can conclude that f is K-quasiregular because $|\bar\partial f/\partial f| \le k$ a.e. by assumption. Since quasiregularity is a local property, it suffices to show that f is quasiregular in an open neighborhood U of each $x \in C$. The assumption allows us to take U so that f maps $U \cap C$ injectively onto a quasi-arc. Then, by composing suitable quasiconformal mappings, we may further assume that U is an open disk centered at $x = 0$ with $U \cap C = U \cap \mathbb{R}$ and that $f(U \cap \mathbb{R}) \subset \mathbb{R}$. Set $U_\pm = \{z \in U; \pm\mathrm{Im}\, z \ge 0\}$. By the reflection principle for quasiregular mappings [?], the mapping $f|_{U_\pm}$ extends to a quasiregular one in U for each signature. This means f is ACL in U, and hence f is quasiregular there. $\square$

By this lemma, our mapping $\hat f$ turns out to be a K-quasiregular mapping on $\widehat{\mathbb{C}}$, that is, $\hat f = g \circ \omega$ for a K-quasiconformal mapping $\omega : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and a rational function g. Suppose that the degree of g is greater than one. Then there exists a branch point, say b, of g.

Without loss of generality, we may assume that $\omega(0)=b$ and $\infty\in C=\partial D$. Furthermore, we may assume that $\eta_0(0)\neq 0$. (If not, consider $1/f$ instead of f in the following.)

Now we recall an important fact on quasiconformal reflections.

Lemma 5 ([?, Lemma I.6.3]). *Let λ be a K-quasiconformal reflection in C with $\infty\in C$. Then*

$$\frac{1}{c(K)}|z-a|\le|\lambda(z)-a|\le c(K)|z-a|$$

for any $z\in\mathbb{C}$ and $a\in C$, where $c(K)>1$ is a constant depending only on K.

Since $\hat f$ is never injective near 0 but $\hat f|_D=f=\eta_1/\eta_0$ is injective near 0, we can select sequences of pairs of points z_n and w_n in D and closed arcs α_n connecting z_n and w_n in D such that $F(z_n)=F(w_n)$ and $F(\alpha_n)$ has winding number 1 around $F(0)=f(0)$, and that z_n,w_n and $\operatorname{diam}\alpha_n$ all tend to 0 as n tends to ∞, where diam stands for the Euclidean diameter.

Now we consider the asymptotic behavior of $F(z)$ as $z\to0$. We have

$$\begin{aligned}F(z)-F(0)&=\frac{\eta_1(z)+(\lambda(z)-z)\eta_1'(z)}{\eta_0(z)+(\lambda(z)-z)\eta_0'(z)}-\frac{\eta_1(0)}{\eta_0(0)}\\&=\frac{(\eta_0(0)\eta_1(z)-\eta_1(0)\eta_0(z))+(\lambda(z)-z)(\eta_0(0)\eta_1'(z)-\eta_1(0)\eta_0'(z))}{\eta_0(0)(\eta_0(z)+(\lambda(z)-z)\eta_0'(z))}.\end{aligned}$$

By the relation $\eta(z)=\eta(0)+z\eta'(0)+O(z^2)$ or similar ones, the numerator in the above can be calculated as

$$z\left(\eta_0(0)\eta_1'(0)-\eta_1(0)\eta_0'(0)\right)+(\lambda(z)-z)\left(1+z\left(\eta_0(0)\eta_1''(0)-\eta_1(0)\eta_0''(0)\right)\right)+O(z^2)$$

when $z\to0$. Noting the relations $\eta_0\eta_1'-\eta_1\eta_0'\equiv1$ and $\eta_j''=-\varphi\eta_j/2$ and Lemma 5, we obtain

$$F(z)-F(0)=\eta_0(0)^{-2}\lambda(z)+O(z^2)\tag{4}$$

as $z\to0$ in D.

Now we may assume $|z_n|\ge|w_n|$ for every n. Since $F(z_n)=F(w_n)$ we have $\delta_n:=|z_n^*-w_n^*|=O(|z_n|^2)$ as $n\to\infty$ by (4), where we set $z_n^*=\lambda(z_n)$ and $w_n^*=\lambda(w_n)$. Similarly, we have $\eta_0(0)^2(F(\alpha_n(t))-F(0))-\alpha_n^*(t)=O(\alpha_n(t)^2)=O(\alpha^*(t)^2)$ as $n\to\infty$, where $\alpha_n^*=\lambda(\alpha_n)$. In particular,

$$|\eta_0(0)^2(F(\alpha_n(t))-F(0))-\alpha_n^*(t)|<|\alpha_n^*(t)|\tag{5}$$

holds for sufficiently large n.

On the other hand, linear connectedness of D^* asserts the existence of a constant $M>1$ such that any pair of points in $D^*\cap B(a,r)$ can be joined by a curve in $D^*\cap B(a,Mr)$ for all $a\in\mathbb{C}$ and $r>0$, where $B(a,r)$ stands for the closed disk centered at a of radius r (see [?] or [?]). In particular, there exists a sequence of curves β_n^* connecting w_n^* and z_n^* in $D^*\cap B(z_n^*,M\delta_n)$. Therefore we have $|\eta_0(0)^2(F(z_n)-F(0))-\beta_n^*(t)|\le M\delta_n+O(|z_n|^2)=O(|z_n|^2)$ as $n\to\infty$, and then

$$|\eta_0(0)^2(F(z_n)-F(0))-\beta_n^*(t)|<|\beta_n^*(t)|\tag{6}$$

for n large enough.

Now we conclude that the closed curves $F(\alpha_n) - F(0)$ and $\gamma_n^* := \alpha_n^* \cdot \beta_n^*$ have the same winding number around 0 for sufficiently large n from (5) and (6). By the choice of α_n, we see that γ_n^* has winding number 1, and hence separates 0 from ∞ for such n. Since $0 \in \partial D$ and γ_n^* is a curve in D^*, D is contained in a bounded component of $\mathbb{C} \setminus \gamma_n^*$. In particular, we have $\operatorname{diam} D \leq \operatorname{diam} \gamma_n^* \leq \operatorname{diam} \alpha_n^* + \operatorname{diam} \beta_n^*$ for sufficiently large n. Since both $\operatorname{diam} \alpha_n^*$ and $\operatorname{diam} \beta_n^*$ tend to 0 as $n \to \infty$, we would have $\operatorname{diam} D = 0$, which is impossible. This contradiction is caused by the assumption $\deg g > 1$. Theorefore we can now conclude that g is a Möbius transformation, and hence the proof of Lemma 3 is complete.

References

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