ON THE TEICHMÜLLER SPACES OF
FUCHSIAN GROUPS OF SCHOTTKY TYPE
AND THE SCHWARZIAN DERIVATIVES
OF UNIVALENT FUNCTIONS

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§1. The main result.

Let $\Gamma$ be a Fuchsian group acting on the upper half plane $\mathbb{H}$. We denote by $B_2(\Gamma)$ the Banach space of all the holomorphic function $\varphi$ on $\mathbb{H}$ which satisfies the functional equation $(\varphi \circ \gamma)(\gamma')^2 = \varphi$ for all $\gamma \in \Gamma$, with finite norm $\|\varphi\| = \sup_{z \in \mathbb{H}} |\varphi(z)|(\text{Im } z)^2$. We shall consider the following subsets of $B_2(\Gamma)$:

$$S(\Gamma) = \{ \varphi \in B_2(\Gamma) : \text{univalent function } f \text{ on } \mathbb{H} \text{ with } S_f = \varphi \},$$

$$T(\Gamma) = \{ S_f \in S(\Gamma) : f \text{ extends to a } (\Gamma\text{-compatible) qc-map of } \bar{\mathbb{C}} \},$$

where $S_f$ denotes the Schwarzian derivative of $f$ defined as follows: $S_f = (f''/f')' - \frac{1}{2}(f''/f')^2$.

It is known that $S(\Gamma)$ is closed and $T(\Gamma)$ is open in $B_2(\Gamma)$. $T(\Gamma)$ is called (the Bers model of) the Teichmüller space of $\Gamma$. It is an interesting problem how near $T(\Gamma)$ is to $S(\Gamma)$. For a cofinite Fuchsian group (i.e., finitely generated Fuchsian group of the first kind) $\Gamma$, the statement $\overline{T(\Gamma)} = S(\Gamma)$ is equivalent to the Bers conjecture: every $B$-group is obtained as a boundary group of Teichmüller space. (This conjecture is still unsolved.)

On the other hand, for any Fuchsian group $\Gamma$ of the second kind, it is known that $\overline{T(\Gamma)} \subseteq S(\Gamma)$ (cf. [G2], [Sug]).

But a weaker statement that $T(\Gamma) = \text{Int} S(\Gamma)$ is proved for some cases ( [G1: $\Gamma = 1$], [Shiga: cofinite $\Gamma$]). The main result of this article is the validity of the
above statement for all Fuchsian groups of Schottky type, where a Fuchsian group $\Gamma$ is called Schottky type in this article, if $\Gamma$ is a Schottky group simultaneously, in other words, $\Gamma$ uniformizes a topologically finite Riemann surface of genus $g$ with $m$ holes, where $m \geq 1$. Also, the Schottky type Fuchsian group can be characterized as the finitely generated, purely hyperbolic Fuchsian group of the second kind.

**Main Theorem.** $\text{Int}S(\Gamma) = T(\Gamma)$ for any Fuchsian group $\Gamma$ of Schottky type.

§2. Sketch of proof.

Let $\Gamma$ be a Fuchsian group of Schottky type. Then, the quotient surface $S_0 = \mathbb{H}/\Gamma$ is a topologically finite Riemann surface of genus $g$ with $m$ holes and its double $S = \Omega(\Gamma)/\Gamma$ is a compact Riemann surface of genus $N = 2g + m - 1$, where $\Omega(\Omega) \subset \hat{\mathbb{C}}$ denotes the region of discontinuity of $\Gamma$. Let $\varphi \in \text{Int}S(\Gamma)$ and $F : \mathbb{H} \rightarrow \hat{\mathbb{C}}$ be a holomorphic map such that $S_F = \varphi$. By the $\Gamma$-automorphy of $\varphi$, $G = F \Gamma F^{-1}$ is a subgroup of Möb which acts on $D = f(\mathbb{H})$. Since $\varphi$ is an interior point of $S(\Gamma)$, it turns out that $G$ is purely loxodromic. Since $G(\cong \Gamma)$ is a free group of finite rank, Maskit’s characterization theorem tells us that $G$ is also a Schottky group of rank $N = 2g + m - 1$. So, the quotient surface $R = \Omega(G)/G$ is a compact genus $N$ surface. Let $p_0 : \Omega(\Gamma) \rightarrow S$ and $p : \Omega(G) \rightarrow R$ be the natural projections. Set $R_0 = p(D) = D/G$, which is isomorphic to $S_0 = \mathbb{H}/\Gamma$ by the conformal map $f$ induced by $F : \mathbb{H} \rightarrow D$. We shall investigate the way of embedding $R_0 \hookrightarrow R$. Now, the proof of Main Theorem devides into several steps.

**Step 1.** $\partial R_0$ consists of mutually disjoint $m$ simple closed curves.

This step needs a localization of Gehring’s method [G1]. In this step, essential is the fact that $\varphi$ is an interior point of $S(\Gamma)$.

**Step 2.** There exists a self-homeomorphism $h$ of $R$ with the following properties:

(i) $h \circ h = \text{id}_R$,

(ii) $h|_{\partial R_0} = \text{id}_{\partial R_0}$,

(iii) $h(R_0) \cap R_0 = \emptyset$, 


(iv) there exists a homeomorphism $H : \Omega(G) \to \Omega(G)$ such that $p \circ H = h \circ p$ on $\Omega(G)$ and that $H = \text{id}_{\partial D \cap \Omega(G)}$.

This step is covered by rather algebraic arguments. For example, the following lemma is utilized.

**Lemma (general property of the normal coverings).**

Suppose that $p : (\Omega, z_0) \to (R, a_0)$ is a normal covering between (connected) pointed manifolds. Let $R_0$ be a subdomain of $R$ such that $a_0 \in R_0$ and $\iota : R_0 \to R$ denote the inclusion map. Then $\iota$ naturally induces the homomorphism $\iota_* : \pi_1(R_0, a_0) \to \pi_1(R, a_0)$. Let $\lambda : \pi(R, a_0) \to G$ be the lifting homomorphism with respect to $z_0$, where $G$ is a covering transformation group of $p : \Omega \to R$. Namely, $g = \lambda[\alpha]$ for $g \in G$ and $[\alpha] \in \pi_1(R, a_0)$ iff the final point of the lift $\tilde{\alpha}$ of $\alpha$ with initial point $z_0$ coincides with $g(z_0)$. Then, the followings hold.

(i) Each component of $p^{-1}(R_0)$ is simply connected $\iff \lambda \circ \iota_*$ is injective.

(ii) $p^{-1}(R_0)$ is connected $\iff \lambda \circ \iota_*$ is surjective.

In particular, if $p^{-1}(R_0)$ is a simply connected domain, then $\iota_* : \pi_1(R_0, a_0) \to \pi_1(R, a_0)$ is an embedding and $\pi_1(R, a_0) = \ker \lambda \rtimes \pi_1(R_0, a_0)$ (semi-direct product).

First of all, we can naturally extend $f$ to a homeomorphism $f : \overline{S_0} \to \overline{R_0}$ by Step 1. Further, by use of Step 2, we can extend $f$ to a homeomorphism $\tilde{f} : S \to R$ in the following way.

$$\tilde{f} = \begin{cases} f & \text{on } \overline{S_0}, \\ h \circ f \circ j & \text{on } S \setminus \overline{S_0}, \end{cases}$$

where $j$ denotes the involution map $S \to S$ induced by conjugation $J(z) = \overline{z}$. By construction, $\tilde{f}$ can be lifted, that is, there exists a homeomorphism $\tilde{F} : \Omega(\Gamma) \to \Omega(G)$ such that $p \circ \tilde{F} = \tilde{f} \circ p_0$. By purely topological arguments, it turns out that $\tilde{F}$ can be naturally extended to a homeomorphism $\tilde{F} : \tilde{C} \to \tilde{C}$. In particular, it is known that $D$ is an image of $\mathbb{H}$ under the self-homeomorphism $\tilde{F}$ of $\tilde{C}$, so $D$ is a Jordan domain.

**Step 3.** $\partial R_0$ is a disjoint union of quasi-analytic curves.
Here, the “quasi-analytic curve” means the quasiconformal image of a circle. For the proof of Step 3, we need just more delicate arguments than in Step 1. By the way, one can prove the following

**Proposition.** Let $S$ and $R$ be compact Riemann surfaces and $S_0 \subset S, R_0 \subset R$ be subdomains with quasi-analytic boundaries. Suppose that $\tilde{f} : S \to R$ is an orientation-preserving homeomorphism such that $\tilde{f}(S_0) = R_0$ and the restriction map $\tilde{f}|_{S_0} : S_0 \to R_0$ is quasiconformal. Then, there exists a quasiconformal map $\tilde{f}_1 : S \to R$ which is homotopic to $\tilde{f}$ and $\tilde{f}_1 = \tilde{f}$ on $R_0$.

By virtue of this proposition, we can choose a quasiconformal $\tilde{f} : S \to R$ as the extension of $f$. Then, a topological extension $\tilde{F} : \hat{C} \to \hat{C}$ of a lift of $\tilde{f}$ is quasiconformal on $\Omega(I')$, so $\tilde{F} : \hat{C} \to \hat{C}$ is a quasiconformal self-homeomorphism since $\Lambda(I') = \hat{C} \setminus \Omega(I') \subset \hat{R}$ is a quasiconformally removable set. Therefore $D = \tilde{F}(\mathbb{H})$ is a quasi-disk, the proof is completed.

**References**


