ON THE TEICHMÜLLER SPACES OF FUCHSIAN GROUPS OF SCHOTTKY TYPE AND THE SCHWARZIAN DERIVATIVES OF UNIVALENT FUNCTIONS

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§1. The main result.

Let Γ be a Fuchsian group acting on the upper half plane $\mathbb{H}$. We denote by $B_2(\Gamma)$ the Banach space of all the holomorphic function φ on $\mathbb{H}$ which satisfies the functional equation $(\varphi \circ \gamma)(\gamma')^2 = \varphi$ for all $\gamma \in \Gamma$, with finite norm $\|\varphi\| = \sup_{z\in\mathbb{H}} |\varphi(z)|(\mathrm{Im}\, z)^2$. We shall consider the following subsets of $B_2(\Gamma)$:

$$S(\Gamma) = \{\varphi \in B_2(\Gamma) : \exists \text{univalent function} f \text{ on } \mathbb{H} \text{ with } S_f = \varphi\},$$

$$T(\Gamma) = \{S_f \in S(\Gamma) : f \text{ extends to a } (\Gamma\text{-compatible}) \text{ qc-map of } \widehat{\mathbf{C}}\},$$

where S_f denotes the Schwarzian derivative of f difined as follows: $S_f = (f''/f')' - \frac{1}{2}(f''/f')^2$.

It is known that $S(\Gamma)$ is closed and $T(\Gamma)$ is open in $B_2(\Gamma)$. $T(\Gamma)$ is called (the Bers model of) the Teichmüller space of Γ. It is an interesting problem how near $T(\Gamma)$ is to $S(\Gamma)$. For a cofinite Fuchsian group (i.e., finitely generated Fuchsian group of the first kind) Γ, the statement $\overline{T(\Gamma)} = S(\Gamma)$ is equivalent to the Bers conjecture: every B-group is obtained as a boundary group of Teichmüller space. (This conjecture is still now unsolved.)

On the other hand, for any Fuchsian group Γ of the second kind, it is known that $\overline{T(\Gamma)} \subsetneqq S(\Gamma)$ (cf. [G2], [Sug]).

But a weaker statement that $T(\Gamma) = \mathrm{Int} S(\Gamma)$ is proved for some cases ([G1: $\Gamma = 1$], [Shiga: cofinite Γ]). The main result of this article is the validity of the

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above statement for all Fuchsian groups of Schottky type, where a Fuchsian group Γ is called Schottky type in this article, if Γ is a Schottky group simultaneously, in other words, Γ uniformizes a topologically finite Riemann surface of genus g with m holes, where $m \geq 1$. Also, the Schottky type Fuchsian group can be characterized as the finitely generated, purely hyperbolic Fuchsian group of the second kind.

Main Theorem. $\mathrm{Int}S(\Gamma) = T(\Gamma)$ *for any Fuchsian group Γ of Schottky type.*

§2. Sketch of proof.

Let Γ be a Fuchsian group of Schottky type. Then, the quotient surface $S_0 = \mathbb{H}/\Gamma$ is a topologically finite Riemann surface of genus g with m holes and its double $S = \Omega(\Gamma)/\Gamma$ is a compact Riemann surface of genus $N = 2g+m-1$, where $\Omega(\Omega) \subset \widehat{\mathbf{C}}$ denotes the region of discontinuity of Γ. Let $\varphi \in \mathrm{Int}S(\Gamma)$ and $F : \mathbb{H} \to \widehat{\mathbf{C}}$ be a holomorphic map such that $S_f = \varphi$. By the Γ-automorphy of φ, $G = F\Gamma F^{-1}$ is a subgroup of Möb which acts on $D = f(\mathbb{H})$. Since φ is an interior point of $S(\Gamma)$, it turns out that G is purely loxodromic. Since $G(\cong \Gamma)$ is a free group of finite rank, Maskit's characterization theorem tells us that G is also a Schottky group of rank $N = 2g + m - 1$. So, the quotient surface $R = \Omega(G)/G$ is a compact genus N surface. Let $p_0 : \Omega(\Gamma) \to S$ and $p : \Omega(G) \to R$ be the natural projections. Set $R_0 = p(D) = D/G$, which is isomorphic to $S_0 = \mathbb{H}/\Gamma$ by the conformal map f induced by $F : \mathbb{H} \to D$. We shall investigate the way of embedding $R_0 \hookrightarrow R$. Now, the proof of Main Theorem devides into several steps.

Step 1. *∂R_0 consists of mutually disjoint m simple closed curves.*

This step needs a localization of Gehring's method [G1]. In this step, essential is the fact that φ is an interior point of $S(\Gamma)$.

Step 2. *There exists a self-homeomorphism h of R with the following properties:*

(i) $h \circ h = \mathrm{id}_R$,

(ii) $h|_{\partial R_0} = \mathrm{id}_{\partial R_0}$,

(iii) $h(R_0) \cap R_0 = \emptyset$,

(iv) there exits a homeomorphism $H : \Omega(G) \to \Omega(G)$ such that $p \circ H = h \circ p$ on $\Omega(G)$ and that $H = \mathrm{id}_{\partial D \cap \Omega(G)}$.

This step is covered by rathar algebraic arguments. For example, the following lemma is utilized.

Lemma (general property of the normal coverings).

Suppose that $p : (\Omega, z_0) \to (R, a_0)$ is a normal covering between (connected) pointed manifolds. Let R_0 be a subdomain of R such that $a_0 \in R_0$ and $\iota : R_0 \to R$ denote the inclusion map. Then ι naturally induces the homomorphism $\iota_* : \pi_1(R_0, a_0) \to \pi_1(R, a_0)$. Let $\lambda : \pi(R, a_0) \to G$ be the lifting homomorphism with respect to z_0, where G is a covering transformation group of $p : \Omega \to R$. Namely, $g = \lambda[\alpha]$ for $g \in G$ and $[\alpha] \in \pi_1(R, a_0)$ iff the final point of the lift $\tilde{\alpha}$ of α with initial point z_0 coincides with $g(z_0)$. Then, the followings hold.

(i) Each component of $p^{-1}(R_0)$ is simply connected $\Longleftrightarrow \lambda \circ \iota_*$ is injective.

(ii) $p^{-1}(R_0)$ is connected $\Longleftrightarrow \lambda \circ \iota_*$ is surjective.

In particular, if $p^{-1}(R_0)$ is a simply connected domain, then $\iota_* : \pi_1(R_0, a_0) \to \pi_1(R, a_0)$ is an embedding and $\pi_1(R, a_0) = \ker \lambda \rtimes \pi_1(R_0, a_0)$ (semi-direct product).

First of all, we can naturally extend f to a homeomorphism $f : \overline{S_0} \to \overline{R_0}$ by Step 1. Further, by use of Step 2, we can extend f to a homeomorphism $\tilde{f} : S \to R$ in the following way.

$$\tilde{f} = \begin{cases} f & \text{on } \overline{S_0}, \\ h \circ f \circ j & \text{on } S \setminus \overline{S_0}, \end{cases}$$

where j denotes the involution map $S \to S$ induced by conjugation $J(z) = \bar{z}$. By construction, $\tilde{f}$ can be lifted, that is, there exists a homeomorphism $\widetilde{F} : \Omega(\Gamma) \to \Omega(G)$ such that $p \circ \widetilde{F} = \tilde{f} \circ p_0$. By purely topological arguments, it turns out that $\widetilde{F}$ can be naturally extended to a homeomorphism $\widetilde{F} : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$. In particular, it is known that D is an image of $\mathbb{H}$ under the self-homeomorphism $\widetilde{F}$ of $\widehat{\mathbf{C}}$, so D is a Jordan domain.

Step 3. *∂R_0 is a disjoint union of quasi-analytic curves.*

Here, the "quasi-analytic curve" means the quasiconformal image of a circle. For the proof of Step 3, we need just more delicate arguments than in Step 1. By the way, one can prove the following

Proposition. *Let S and R be compact Riemann surfaces and $S_0 \subset S, R_0 \subset R$ be subdomains with quasi-analytic boundaries. Suppose that $\tilde{f} : S \to R$ is an orientation-preserving homeomorphism such that $\tilde{f}(S_0) = R_0$ and the restriction map $\tilde{f}|_{S_0} : S_0 \to R_0$ is quasiconformal. Then, there exists a quasiconformal map $\tilde{f}_1 : S \to R$ which is homotopic to $\tilde{f}$ and $\tilde{f}_1 = \tilde{f}$ on R_0.*

By virture of this proposition, we can choose a quasiconformal $\tilde{f} : S \to R$ as the extension of f. Then, a topological extension $\widetilde{F} : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ of a lift of $\tilde{f}$ is quasiconformal on $\Omega(\Gamma)$, so $\widetilde{F} : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ is a quasiconformal self-homeomorphism since $\Lambda(\Gamma) = \widehat{\mathbf{C}} \setminus \Omega(\Gamma) \subset \widehat{\mathbb{R}}$ is a quasiconformally removable set. Therefore $D = \widetilde{F}(\mathbb{H})$ is a quasi-disk, the proof is completed.

References

[G1] F.[redacted]V. Gehring, *Univalent functions and the universal Teichmüller space*, Comment. Math. Helv. **52** (1977), 561–572.

[G2] F.[redacted]V. Gehring, *Spirals and the universal Teichmüller space*, Acta Math. **141** (1978), 99–113.

[Shiga] H. Sh[redacted]ga, *Characterization of quasi-disks and Teichmüller spaces*, Tôhoku Math. J. **37** (1985), 541–552.

[Sug] T. [redacted]ugawa, *On the Bers conjecture for Fuchsian groups of the second kind*, J. Math. Kyoto Univ. **32** (1992), 45–52.