ON THE SCHLICHT PROJECTIVE STRUCTURES ON
COMPACT BORDERED RIEMANN SURFACES WITH

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ABSTRACT. In the present session board, the main result in [Sug] is explained, that is, \(\text{Int}(S(\Gamma)) = T(\Gamma)\) for any Fuchsian group uniformizing a compact bordered Riemann surface with nonempty boundary, i.e., for any finitely generated, purely hyperbolic Fuchsian group of the second kind, where \(S(\Gamma)\) denotes the Schwarzian derivatives of all the \(\Gamma\)-equivariant schlicht holomorphic functions and \(T(\Gamma)\) is the (Bers model of) Teichmüller space of \(\Gamma\). The paper [Sug] of the author also contains some results concerning with \(\text{Int}(S(\Gamma))\) for general Fuchsian groups \(\Gamma\).

§1. Introduction

Let \(\Gamma\) be an arbitrary Fuchsian group acting on the upper half plane \(\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im} z > 0\}\) and \(Q(\mathbb{H}, \Gamma)\) be the space consisting all the holomorphic quadratic differential \(\varphi\) on \(\mathbb{H}\) for \(\Gamma\), i.e., \(\varphi\) is holomorphic function on \(\mathbb{H}\) satisfying a condition \(\varphi \circ \gamma(\gamma')^2 = \varphi\) for \(\gamma \in \Gamma\). We denote by \(S(\Gamma)\) the set consisting of the Schwarzian derivative \(S_f\) of all the univalent meromorphic functions \(f\) on \(\mathbb{H}\) with \(f \circ \gamma = \chi(\gamma) \circ f\) on \(\mathbb{H}\) for some group homomorphism \(\chi : \Gamma \rightarrow \text{Möb}\). Then, by the Kraus-Nehari theorem, it turns out that \(S(\Gamma)\) is a bounded closed subset of the complex Banach space \(B_2(\mathbb{H}, \Gamma) = \{\varphi \in Q(\mathbb{H}, \Gamma) \mid \|\varphi\|_\mathbb{H} < \infty, \text{where } \|\varphi\|_\mathbb{H} = \sup_{z \in \mathbb{H}} (2\text{Im} z)^2 |\varphi(z)|\}.

Furthermore, we define a subset \(T(\Gamma)\) of \(S(\Gamma)\) by \(\{\varphi = S_f \in S(\Gamma) \mid f(\mathbb{H}) \) is a quasidisk \}. This subset \(T(\Gamma)\) is called (the Bers model of) the Teichmüller space of the Fuchsian group \(\Gamma\), and is known to be a bounded connected open subset of \(B_2(\mathbb{H}, \Gamma)\).

It is an interesting topic to investigate how the Teichmüller space \(T(\Gamma)\) is embedded in \(S(\Gamma)\). Generally, \(\overline{T(\Gamma)} \subsetneq S(\Gamma)\) holds. In fact, first Gehring has shown that \(\overline{T(1)} \subsetneq S(1)\) in [7], and later the author proved in [14] that \(\overline{T(\Gamma)} \subsetneq S(\Gamma)\) for any Fuchsian group \(\Gamma\) of the second kind. Moreover, recently K. Matsuzaki showed in [9] the existence of certain infinitely generated Fuchsian groups \(\Gamma\) of the first kind such that \(\overline{T(\Gamma)} \subsetneq S(\Gamma)\). But, it is still a difficult problem to decide whether \(\overline{T(\Gamma)} = S(\Gamma)\) for a finitely generated Fuchsian group \(\Gamma\) of the first kind.

On the other hand, Gehring has shown in [6] that \(\text{Int}(S(1)) = T(1)\). Furthermore Úzuraev showed in [17] that \(T(\Gamma)\) is the zero component of \(\text{Int}(S(\Gamma))\) for an arbitrary Fuchsian group \(\Gamma\). Thus, it is naturally conjectured that \(\text{Int}(S(\Gamma)) = T(\Gamma)\) for any \(\Gamma\). In this direction, Shiga proved in [13] that the above conjecture holds if \(\Gamma\) is finitely generated Fuchsian group of the first kind, equivalently, if \(B_2(\mathbb{H}, \Gamma)\) is finite dimensional.

The main result in this article is the following.

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Theorem 1.
If \( \Gamma \) is a finitely generated, purely hyperbolic Fuchsian group of the second kind, then \( T(\Gamma) = \text{Int}S(\Gamma) \).

Remark.
For a Fuchsian group \( \Gamma \) acting on \( \mathbb{H} \), the following conditions are mutually equivalent:

(i) \( \Gamma \) is finitely generated, purely hyperbolic and of the second kind,

(ii) \( \Gamma \) is a Schottky group,

(iii) \( \Gamma \) is a uniformizing group of a compact bordered Riemann surface with nonempty boundary, more precisely, \( \Gamma \) is the covering transformation group of a holomorphic universal covering \( p : \mathbb{H} \to R \), where \( R \) is a compact Riemann surface of genus \( g(\geq 0) \) with mutually closed topological disks \( \mathcal{D}_1, \ldots, \mathcal{D}_m \) removed \( (m \geq 1) \).

In case of (iii), we say that \( R \) is of conformal type \( (g, 0, m) \), and we should note that \( \Gamma \) is a free group of rank \( 2g + m - 1 \).

\[ \S 2. \text{Proof of the main theorem} \]

In the beginning, we consider a general Fuchsian group \( \Gamma \) of the second kind. Let \( \varphi = S_f \in \text{Int}S(\Gamma) \) and set \( D = f(\mathbb{H}) \). We denote \( \chi : \Gamma \to \text{Möb} \) the monodromy homorphism for \( f \), that is, \( f \circ \gamma = \chi(\gamma) \circ f \) for \( \gamma \in \Gamma \). First, it is easy to see that \( \chi \) is type preserving monomorphism (cf. [16] or [Sug]) and \( G = \chi(\Gamma) \) is a Kleinian group acting on \( D \).

Essentially, it is sufficient to prove that \( D \) is a quasidisk. Now, we recall a characterization of the quasidisk due to Gehring [G1]: a bounded simply connected domain \( D \) is a quasidisk if and only if there exists a constant \( A > 1 \) such that

1. for an arbitrary disk \( \Delta \), any two points in \( D \cap \Delta \) can be connected by a path in \( D \cap \Delta_A \), and

2. for an arbitrary disk \( \Delta \), any two points in \( D \setminus \Delta_A \) can be connected by a path in \( D \setminus \Delta \), where \( \Delta_A \) denotes \( \{|z - z_0| < Ar\} \) if \( \Delta = \{|z - z_0| < r\} \).

First, in the line along the above characterization, we can show the following propositions.

Proposition 1.
Suppose that a Kleinian group \( G \) acts on a simply connected plane domain \( D \subset \mathbb{C} \) of hyperbolic type. And suppose that \( D \) is a \( G \)-Schwarzian domain with constant \( \varepsilon > 0 \), the following is valid for an appropriate constant \( B > 1 \) depending only on \( \varepsilon \): for an arbitrary \( \Delta \in \mathcal{D}_B(\Omega(G)) \) such that \( p|_{\Delta_B} \) is injective, any two points in \( \Delta \cap D \) can be joined by a path in \( \Delta_B \cap D \).

References


