ON THE SCHLICHT PROJECTIVE STRUCTURES ON COMPACT BORDERED RIEMANN SURFACES WITH

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Abstract. In the present session board, the main result in [Sug] is explained, that is, $\mathrm{Int}S(\Gamma) = T(\Gamma)$ for any Fuchsian group uniformizing a compact bordered Riemann surface with nonempty boundary, i.e., for any finitely generated, purely hyperbolic Fuchsian group of the second kind, where $S(\Gamma)$ denotes the Schwarzian derivatives of all the Γ-equivariant schlicht holomorphic functions and $T(\Gamma)$ is the (Bers model of) Teichmüller space of Γ. The paper [Sug] of the author also contains some results concerning with $\mathrm{Int}S(\Gamma)$ for general Fuchsian groups Γ.

§1. Introduction

Let Γ be an arbitrary Fuchsian group acting on the upper half plane $\mathbb{H} = \{z \in \mathbb{C}; \mathrm{Im} z > 0\}$ and $Q(\mathbb{H}, \Gamma)$ be the space consisting all the holomorphic quadratic differntial φ on $\mathbb{H}$ for Γ, i.e., φ is holomorphic function on $\mathbb{H}$ satisfying a condtion $\varphi \circ \gamma(\gamma')^2 = \varphi$ for $\gamma \in \Gamma$. We donote by $S(\Gamma)$ the set consisting of the Schwarzian derivative S_f of all the univalent meromorphic functions f on $\mathbb{H}$ with $f \circ \gamma = \chi(\gamma) \circ f$ on $\mathbb{H}$ for some group homomorphism $\chi : \Gamma \to \text{Möb}$. Then, by the Kraus-Nehari theorem, it turns out that $S(\Gamma)$ is a bounded closed subset of the complex Banach space $B_2(\mathbb{H}, \Gamma) = \{\varphi \in Q(\mathbb{H}, \Gamma) | \|\varphi\|_{\mathbb{H}} < \infty$, where $\|\varphi\|_{\mathbb{H}} = \sup_{z \in \mathbb{H}} (2\mathrm{Im} z)^2 |\varphi(z)|$.

Furthermore, we define a subset $T(\Gamma)$ of $S(\Gamma)$ by $\{\varphi = S_f \in S(\Gamma) | f(\mathbb{H})$ is a quasidisk $\}$. This subset $T(\Gamma)$ is called (the Bers model of) the Teichmüller space of the Fuchsian group Γ, and is known to be a bounded connected open subset of $B_2(\mathbb{H}, \Gamma)$.

It is an intersting topic to investigate how the Teichmüller space $T(\Gamma)$ is embedded in $S(\Gamma)$. Generally, $\overline{T(\Gamma)} \subsetneqq S(\Gamma)$ holds. In fact, first Gehring has shown that $\overline{T(1)} \subsetneqq S(1)$ in [7], and later the author proved in [14] that $\overline{T(\Gamma)} \subsetneqq S(\Gamma)$ for any Fuchsian group Γ of the second kind. Moreover, recently K. Matsuzaki showed in [9] the existence of certain infinitely generated Fuchsian groups Γ of the first kind such that $\overline{T(\Gamma)} \subsetneqq S(\Gamma)$. But, it is still a difficult problem to decide whether $\overline{T(\Gamma)} = S(\Gamma)$ for a finitely generated Fuchsian group Γ of the first kind.

On the other hand, Gehring has shown in [6] that $\mathrm{Int}S(1) = T(1)$. Furthermore Žuravlev showed in [17] that $T(\Gamma)$ is the zero component of $\mathrm{Int}S(\Gamma)$ for an arbitrary Fuchsian group Γ. Thus, it is naturally conjectured that $\mathrm{Int}S(\Gamma) = T(\Gamma)$ for any Γ. In this direction, Shiga proved in [13] that the above conjecture holds if Γ is finitely generated Fuchsian group of the first kind, equivalently, if $B_2(\mathbb{H}, \Gamma)$ is finite dimensional.

The main result in this article is the following

Theorem 1.

If Γ is a finitely generated, purely hyperbolic Fuchsian group of the second kind, then $T(\Gamma) = \mathrm{Int}S(\Gamma)$.

Remark.

For a Fuchsian group Γ acting on $\mathbb{H}$, the following conditions are mutually equivalent:

(i) Γ is finitely generated, purely hyperbolic and of the second kind,
(ii) Γ is a Schottky group,
(iii) Γ is a uniformizing group of a compact bordered Riemann surface with nonempty boundary, more presicely, Γ is the covering transformation group of a holomorphic universal covering $p : \mathbb{H} \to R$, where R is a compact Riemann surface of genus $g(\geq 0)$ with mutually closed topological disks $\overline{D}_1, \ldots, \overline{D}_m$ removed $(m \geq 1)$.

In case of (iii), we say that R is of conformal type $(g, 0, m)$, and we should note that Γ is a free group of rank $2g + m - 1$.

§2. Proof of the main theorem

In the beginning, we consider a general Fuchsian group Γ of the second kind. Let $\varphi = S_f \in \mathrm{Int}S(\Gamma)$ and set $D = f(\mathbb{H})$. We denote $\chi : \Gamma \to$ Möb the monodromy homorphism for f, that is, $f \circ \gamma = \chi(\gamma) \circ f$ for $\gamma \in \Gamma$. First, it is easy to see that χ is type preserving monomorphism (cf. [16] or [Sug]) and $G = \chi(\Gamma)$ is a Kleinian group acting on D.

Essentially, it is sufficient to prove that D is a quasidisk. Now, we recall a characterization of the quasidisk due to Gehring [G1]: a bounded simply connected domain D is a quasidisk if and only if there exists a constant $A > 1$ such that

(1) for an arbitrary disk Δ, any two points in $D \cap \Delta$ can be connected by a path in $D \cap \Delta_A$, and
(2) for an arbitrary disk Δ, any two points in $D \setminus \Delta_A$ can be connected by a path in $D \setminus \Delta$, where Δ_A denotes $\{|z - z_0| < Ar\}$ if $\Delta = \{|z - z_0| < r\}$.

First, in the line along the above characterization, we can show the following propositions.

Propostion 1.

Suppose that a Kleinian group G acts on a simply connected plane domain $D \subset \mathbb{C}$ of hyperbolic type. And suppose that D is a G-Schwarzian domain with constant $\varepsilon > 0$, *the following is valid for an appropriate constant* $B > 1$ *depending only on* ε *: for an arbitrary $\Delta \in \mathcal{D}_B(\Omega(G))$ such that $p|_{\Delta_B}$ is injective, any two points in $\Delta \cap D$ can be joined by a path in $\Delta_B \cap D$.*

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