EXPLICIT ESTIMATES OF UNIFORM PERFECTNESS AND ITS APPLICATIONS

TOSHIYUKI SUGAWA

1. Definitions and notations

Let R be a hyperbolic Riemann surface, i.e. there exists a holomorphic universal covering map $f : \Delta \to R$ from the unit disk Δ onto R with the covering transformation group Γ, where Γ is a torsion-free Fuchsian group. Since the Poincaré metric $\rho_\Delta = \frac{|dz|}{1-|z|^2}$ of Δ is invariant under the action of $\Gamma \subset \mathrm{Möb}(\Delta)$, it may be projected to the metric $\rho_R = \rho_R(z)|dz|$ on R, which is also called the Poincaré (or hyperbolic) metric of R. We denote by $\iota_R(q)$ the injectivity radius of R at $q \in R$ with respect to the hyperbolic metric of R, and set $I_R = \inf_{q\in R} \iota_R(q)$. Since the sectional curvature of R is of negative constant -4, we may say that R is *of bounded geometry* if $I_R > 0$. And we denote by $\mathcal{C}_R$ the set of free homotopy classes $[\alpha]$ of essential loops α in R. We define the hyperbolic length $\ell_R[\alpha]$ of $[\alpha]$ by the infimum of $\ell_R(\alpha') = \int_{\alpha'} \rho_R(z)|dz|$, where α' ranges over all loops freely homotopic to α in R.

We remark here that $2I_R = \inf_{\alpha\in\mathcal{C}_R} \ell_R[\alpha] = \inf_{\gamma\in\Gamma\setminus\{1\}} \cosh^{-1}(|\mathrm{tr}\gamma|/2)$.

Now we define the modulus of R. A ring domain $A \subset R$ is said to be essential if the core curve of it is essential in R. The modulus $m(A)$ of A is defined as the number m such that A is conformally equivalent to the round annulus $\{z \in \mathbb{C}; 1 < |z| < e^m\}$. (Note that this definition may be different from standard one.) We set $M_R = \inf m(A)$, where the infimum is taken over all essential ring domains A in R, and call it the *modulus* of R. By the following estimate, we know that $M_R < \infty$ if and only if R is of bounded geometry.

Theorem 1 (cf. [12]). *Let R be a hyperbolic Riemann surface with the hyperbolic metric ρ_R of constant curvature -4. Then, we have the following estimate.*

$$2I_R \le \frac{\pi^2}{M_R} \le \min\{2I_R e^{2I_R}, 2{I_R}^2 \coth^2 I_R\}. \tag{1}$$

Here, the equality occurs in the left-hand side if and only if R is a doubly connected planar Riemann surface or $I_R = 0$.

By a numerical calculation, we see that $2I_R e^{2I_R} > 2I_R^2 \coth^2 I_R$ if and only if $I_R > 0.45752\cdots$. The above estimate partly follows from the next comparison theorem, which is an improvement of Maskit's result [7].

Theorem 2 ([12]). *For any non-contractible simple closed curve α in R, we have an estimate:*

$$\frac{2}{\pi}\ell_R[\alpha] \leq E_R[\alpha] \leq \frac{2}{\pi}\ell_R[\alpha]e^{\ell_R[\alpha]},$$

where $E_R[\alpha]$ denotes the extremal length of $[\alpha]$.

In particular, when R is a plane domain, R is of bounded geometry if and only if ∂R is uniformly perfect, i.e. there exists a constant $0 < c < 1$ such that $\partial R \cap \{z; cr < |z-a| < r\} \neq \emptyset$ for any $a \in \partial R$ and $0 < r < \operatorname{diam}\partial R$, and also so many equivalent conditions for uniform perfectness are known. See, for example, [8], [9] and [12] and its references.

2. Hausdorff dimension

Järvi and Vuorinen showed in their paper [4] that a uniformly perfect set is always of positive Hausdorff dimension (depending only on its uniform perfectness constant). More precisely, we can formulate this as below.

Theorem 3 ([12]). *Let E be a uniformly perfect set in $\widehat{\mathbb{C}}$ and M° denote the supremum of moduli of essential round annuli in $\mathbb{C} \setminus E$. Then we have the following estimate.*

$$\text{H-dim}E \geq \frac{\log 2}{\log(2e^{M^\circ}+1)} \left(\geq \frac{\log 2}{M^\circ + \log 3}\right). \tag{2}$$

We note here that $M^\circ \leq \sup_D M_D$, where D ranges over all the connected components of $\mathbb{C} \setminus E$.

3. Isoperimetric inequality

In this section, we shall consider the hyperbolic isoperimetric inequality for a hyperbolic Riemann surface R. Let $\mathcal{D}_R$ denote the set of relatively compact subdomains of R with smooth boundary. We will say that R satisfies the hyperbolic isoperimetric inequality if $h(R) := \sup_{D\in\mathcal{D}_R} \frac{A_R(D)}{\ell_R(\partial D)} < \infty$, where $A_R(D)$ is the hyperbolic area $\iint_D \rho_R(z)^2 dxdy$ of D and $\ell_R(\partial D)$ is the hyperbolic length of ∂D in R.

The isoperimetric constant $h(R)$ is important at the connection with the bottom $b(R)$ of the spectrum of the Laplace-Beltrami operator on R with respect to the hyperbolic metric. In fact, it is known that $1/16h(R)^2 \leq b(R) \leq 3/4h(R)$, where the lefthand side is called Cheeger's inequality. And $b(R)$ relates the exponent $\delta(R)$ of convergence of R. The following is known as the theorem of Elstrodt, Patterson and Sullivan:

$$b(R) = \begin{cases} \frac{1}{4} & \text{if } 0 \leq \delta(R) \leq \frac{1}{2}, \\ \delta(R)(1-\delta(R)) & \text{if } \frac{1}{2} \leq \delta(R) \leq 1. \end{cases}$$

Fernández and Rodríguez ([1] and [2]) showed that a hyperbolic plane domain of bounded geometry satisfies the hyperbolic isoperimetric inequality. We would like to generalize this result to the case of finite genus. Compact hyperbolic Riemann surfaces are always of bounded geometry but do not satisfy the hyperbolic isoperimetric inequality, so we should exclude these cases. A non-compact hyperbolic Riemann surface of bounded geometry and of finite genus g is roughly isometric to a plane domain of bounded geometry in the sense of Kanai [5], thus we know that R satisfies also the hyperbolic isoperimetric inequality, but, of course, it is difficult to obtain explicit bound for $h(R)$ by this method.

We now present a concrete estimate of $h(R)$ as follows. In the proof, we employ the idea of M. Suzuki [13] as is indicated in [1].

Theorem 4. *Let R be a non-compact Riemann surface of bounded geometry and of finite genus g. Then the isoperimetric constant $h(R)$ can be estimated as*

$$h(R) \leq 1 + \frac{(2g+1)\pi}{2I_R}.$$

In particular, $b(R) > 0$ and $\delta(R) < 1$.

4. Application to rational maps and Kleinian groups

The uniform perfectness of the Julia set of a rational map of degree > 1 was established by Mañé-da Rocha [6] and Hinkkanen [3]. But, their proofs were done by contradiction, so no explicit bounds for unifom perfectness were given. Here, we shall exhibit an explicit one. For more detailed account, see [10].

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d > 1$ and $\mathrm{Crit}(f)$ the set of critical points of f in the Fatou set $\Omega = \widehat{\mathbb{C}} \setminus J_f$. We note that $\#\mathrm{Crit}(f) \leq 2d-2$. Put

$$C_1 = \min_{v_1 \neq v_2 \in f(\mathrm{Crit}(f))} d_\Omega(v_1, v_2), \quad C_2 = \min_{v \in f(\mathrm{Crit}(f))} 2\iota_\Omega(v),$$

where d_Ω and ι_Ω denote the hyperbolic distance and the injectivity radius in Ω, respectively. And let $A_1, \cdots, A_t$ be the complete system of representatives of the cycles of Herman rings of f. We note here that, by Shishikura's theorem, $0 \leq t \leq d-2$, in particular, if $d = 2$ there are no Herman rings. Put $C_3 = \min\{I_{A_1}, \cdots, I_{A_t}\}$, then we have the next

Theorem 5. *For an arbitrary rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d > 1$, the following holds.*

$$I_{\Omega_f} \geq \min\{C_1, C_2, C_3\}.$$

In particular, the Julia set J_f of f is uniformly perfect.

Actually, the above theorem follows from a somewhat stronger result [10].

Next, we shall consider a non-elementary Kleinian groups G with non-empty region of discontinuity $\Omega(G)$. For simplicity, we assume that G is torsion-free. Then, the quotient space $R = \Omega(G)/G$ becomes a (countable disjoint union of) hyperbolic Riemann surface(s). We set $L_R^* = \inf_{[\alpha]} \ell_R[\alpha]$, where the infimum is taken over all $[\alpha] \in \mathcal{C}_R$ with $\ell_R[\alpha] > 0$, and R will be called *of Lehner type* if $L_R^* > 0$.

It is known that G is finitely generated then the limit set $\Lambda(G)$ is uniformly perfect (see, [9]). We can generalize this to the following form. (For more general and stronger statement, we refer to [11].)

Theorem 6. *If a torsion-free, non-elementary Kleinian group G has the quotient $\Omega(G)/G$ of Lehner type, then the limit set $\Lambda(G)$ is uniformly perfect. Moreover, $2I_{\Omega(G)} \geq L^*_{\Omega(G)/G}$.*

References

1. Fernández, J. L. Domains with strong barrier, *Rev. Mat. Iberoamericana*, **5** (1989), 47–65.
2. Fernández, J. L. and Rodríguez, J. M. The exponent of convergence of Riemann surfaces. Bass Riemann surfaces, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **15** (1990), 165–183.
3. Hinkkanen, A. Julia sets of rational functions are uniformly perfect, *Math. Proc. Cambridge Philos. Soc.*, **113** (1993), 543–559.
4. Järvi, P. and Vuorinen, M. Uniformly perfect sets and quasiregular mappings, *J. London Math. Soc.*, **54** (1996), 515–529.
5. Kanai, M. Rough isometries, and combinatorial approximations of geometries of noncompact Riemannian manifolds, *J. Math. Soc. Japan*, **37** (1985), 391–413.
6. Mañé, R. and da Rocha, L. F. Julia sets are uniformly perfect, *Proc. Amer. Math. Soc.*, **116** (1992), 251–257.
7. Maskit, B. Comparison of hyperbolic and extremal lengths, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **10** (1985), 381–386.
8. Pommerenke, C. Uniformly perfect sets and the Poincaré metric, *Ark. Math.*, **32** (1979), 192–199.
9. Pommerenke, C. On uniformly perfect sets and Fuchsian groups, *Analysis*, **4** (1984), 299–321.
10. Sugawa, T. An explicit bound for uniform perfectness of the Julia sets of rational maps, Preprint (1997).
11. Sugawa, T. Uniform perfectness of the limit sets of Kleinian groups, Preprint (1997).
12. Sugawa, T. Various domain constants related to uniform perfectness, Preprint (1997).
13. Suzuki, M. Comportement des applications holomorphes autour d'un ensemble polaire, *C. R. Acad. Sci. Paris Sér. I Math.*, **304** (1987), 191–194.

Department of Mathematics, Kyoto University, 606-01 Sakyo-ku, Kyoto City, Japan

E-mail address: sugawa@kusm.kyoto-u.ac.jp