HOLOMORPHIC MOTION AND QUASICONFORMAL EXTENSION

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In this talk, we present applications of holomorphic motions to the quasiconformal extension of univalent functions. Compare the following two results.

Theorem A (Krzyż [2]). *Let ω be an analytic function on the unit disk Δ with $|\omega'(z)| \le k$, where $0 \le k < 1$ is a constant. Then the function $f(z) = z + \omega(1/z)$ on the outside of Δ can be extended to a k-quasiconformal automorphism of the Riemann sphere.*

Theorem B (Fait, Krzyż and Zygmunt [1, Theorem 2']). *Let ω be an analytic function on the unit disk Δ with $|\omega'(z)| \le k$, where $0 \le k < 1$ is a constant. Then the function $f(z) = z+\omega(z)$ on Δ can be extended to a k-quasiconformal automorphism of the Riemann sphere.*

As the function $f(z) = z + k/z$ shows, Theorem A is best possible, however Theorem B can be improved as follows: *Under the same hypothesis as above, the function $f(z) = z + \omega(z)$ can be extended to a k'-quasiconformal automorphism of the Riemann sphere, where $k' = k/(2 - k)$.* In fact, this can be easily obtained as a corollary of the following result, which is one of our results presented here.

Theorem. *Let k be a constant in $[0, 1)$. For a normalized analytic function $f(z) = z + a_2z^2 + \dots$ on the unit disk Δ, let $p(z)$ represent one of the quantities $zf'(z)/f(z)$, $1 + zf''(z)/f'(z)$ or $f'(z)$. If $p(\Delta) \subset \{w \in \mathbb{C}; |w - (1 + k^2)/(1 - k^2)| \le 2k/(1 - k^2)\}$, the function f can be extended to a k-quasiconformal automorphism of the Riemann sphere.*

In the case that $p(z) = zf'(z)/f(z)$ the above result is best possible as the function $p(z) = (1 + kz^2)/(1 - kz^2)$ indicates.

The above theorem can be shown by means of the optimal λ-lemma first proved by Słodkowski [3]. We will present a general principle which enables us to deduce quasiconformal extension criteria from only univalence criteria.

References

[1] M. Fait, J. G. Krzyż, and J. Zygmunt, *Explicit quasiconformal extensions for some classes of univalent functions*, Comment. Math. Helv. **51** (1976), 279–285.

[2] J. G. Krzyż, *Convolution and quasiconformal extension*, Comment. Math. Helvetici **51** (1976), 99–104.

[3] Z. Słodkowski, *Holomorphic motions and polynomial hulls*, Proc. Amer. Math. Soc. **111** (1991), 347–355.

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