A REMARK ON AHLFORS' UNIVALENCE CRITERION

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Abstract. In this talk, we will remove an additional assumption made for Ahlfors' univalence criterion. This leads to an estimate of the inner radius of univalence for an arbitrary quasidisk in terms of a quasiconformal reflection.

Summary

Let D be a domain in the Riemann sphere $\widehat{\mathbb{C}}$ with the hyperbolic metric $\rho_D(z)|dz|$ of constant negative curvature -4. For a holomorphic function φ on D, we define the hyperbolic sup-norm of φ by

$$\|\varphi\|_D = \sup_{z\in D} \rho_D(z)^{-2}|\varphi(z)|.$$

We denote by $B_2(D)$ the complex Banach space consisting of all holomorphic functions of finite hyperbolic sup-norm. As is well known, the Schwarzian derivative $S_f = (f''/f')' - (f''/f')^2/2$ of a univalent function on D satisfies $\|S_f\|_D \le 12$ (see [3]). This result is classical for the unit disk $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$, actually, a better estimate $\|S_f\|_{\mathbb{D}} \le 6$ holds. On the other hand, Nehari's theorem asserts that if a locally univalent function f on $\mathbb{D}$ satisfies $\|S_f\|_{\mathbb{D}} \le 2$, then f is necessarily univalent. Hille's example [7] shows that the number 2 is best possible. We now define the quantity $\sigma(D)$, which is called the *inner radius of univalence* of D, as the infimum of the norm $\|S_f\|_D$ of those locally univalent meromorphic function f on D which are not globally univalent. In other words, $\sigma(D)$ is the possible largest value $\sigma \ge 0$ with the property that the condition $\|S_f\|_D \le \sigma$ implies univalence of f in D. In the case $D = \mathbb{D}$, we already know $\sigma(\mathbb{D}) = 2$. For a comprehensive exposition of these notions and some background, we refer the reader to the book [9] of O. Lehto.

Ahlfors [1] showed that every quasidisk has positive inner radius of univalence. Conversely, Gehring [6] proved that if a simply connected domain has positive inner radius of univalence then it must be a quasidisk. Later, Lehto [8] pointed out the inner radius of univalence of a quasidisk can be estimated by the Ahlfors method as

$$\sigma(D) \ge 2 \inf_{z\in D'} \frac{|\bar\partial\lambda(z)| - |\partial\lambda(z)|}{|\lambda(z) - z|^2\rho_D(z)^2}, \tag{1}$$

where λ is a quasiconformal reflection in ∂D which is continuously differentiable off ∂D and $D' = D \setminus \{\infty, \lambda(\infty)\}$. However, in order to obtain the estimate (1) rigorously, a kind of approximation procedure must work, so an additional assumption was needed. For example, Lehto [9, Lemma III.5.1] assumed the quasidisk D to be exhausted by domains of the form $\{rz; z \in D\}$ for $0 < r < 1$. More recently, Betker [5] gave a similar result

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for general quasidisks under the assumption that the quasiconformal reflection λ is of a special form associated with the Löwner chains.

Remark that if we content ourselves with an estimate of the form $\sigma(D) \geq C(K)$ for a K-quasidisk D, where $C(K)$ is a positive constant depending only on K, the original idea of Ahlfors [1] is sufficient (see also [2, Chapter VI] and [9, Theorem II.4.1]).

Our main result is to show (1) without any additional assumption, which might be known as a kind of folklore.

Theorem 1. *Let D be a quasidisk with a quasiconformal reflection λ in ∂D which is continuously differentiable off ∂D. Then the inequality* (1) *holds for D.*

We shall show this theorem by usual normal family argument and in the course of proof we make essential use of the following result due to Bers [4, Lemma 1].

Proposition 2. *Let D be a Jordan domain in $\widehat{\mathbb{C}}$. For any $\varphi \in B_2(D)$ there exists a sequence $(\varphi_j)_j$ of holomorphic functions in $\overline{D}$ such that $\|\varphi_j\|_D \leq \|\varphi\|_D$ and φ_j tends to φ uniformly on each compact subset of D as $j \to \infty$.*

By this result, we have only to handle a $\varphi \in B_2(D)$ holomorphic in $\overline{D}$. But some difficulty still remains, which can be overcome by a topological argument involving linear connectedness of quasidisks and a Painlevé type theorem for quasiregular mappings. We shall give the details a little bit more in the talk.

References

1. L. V. Ahlfors, *Quasiconformal reflections*, Acta Math. **109** (1963), 291–301.
2. ———, *Lectures on Quasiconformal Mappings*, van Nostrand, 1966.
3. A. F. Beardon and F. W. Gehring, *Schwarzian derivatives, the Poincaré metric and the kernel function*, Comment. Math. Helv. **55** (1980), 50–64.
4. L. Bers, *A non-standard integral equation with applications to quasiconformal mappings*, Acta Math. **116** (1966), 113–134.
5. Th. Betker, *Univalence criteria and Löwner chains*, Bull. London Math. Soc. **23** (1991), 563–567.
6. F. W. Gehring, *Univalent functions and the Schwarzian derivative*, Comment. Math. Helv. **52** (1977), 561–572.
7. E. Hille, *Remarks on a paper by Zeev Nehari*, Bull. Amer. Math. Soc. **55** (1949), 552–553.
8. O. Lehto, *Remarks on Nehari's theorem about the Schwarzian derivative and schlicht functions*, J. Analyse Math. **36** (1979), 184–190.
9. ———, *Univalent Functions and Teichmüller Spaces*, Springer-Verlag, 1987.

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