GROWTH AND COEFFICIENT ESTIMATES FOR UNIFORMLY
LOCALY UNIVALENT FUNCTIONS ON THE UNIT DISK

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The present article is a summary of our paper [4] which will appear somewhere.
We will call a holomorphic function \( f \) on the unit disk \( \mathbb{D} \) uniformly locally univalent if \( f \) is univalent on each hyperbolic disk \( D(a, \rho) = \{ z \in \mathbb{D}; \left| \frac{z-a}{1-\overline{a}z} \right| < \tanh \rho \} \) with radius \( \rho \) and center \( a \in \mathbb{D} \) for a positive constant \( \rho \). It is well-known (cf. [7]) that a holomorphic function \( f \) on the unit disk is uniformly locally univalent if and only if the pre-Schwarzian derivative (or nonlinearity) \( T_f = f''/f' \) of \( f \) is hyperbolically bounded, i.e., the norm
\[
\|T_f\| = \sup_{z \in \mathbb{D}} (1 - |z|^2) |T_f(z)|
\]
is finite. This quantity can be regarded as the Bloch norm of the function \( \log f' \).

Because \( T_f \) is invariant under the post-composition by a non-constant linear function, we may assume that a holomorphic function \( f \) on the unit disk is normalized so that \( f(0) = 0 \) and \( f'(0) = 1 \). We denote by \( \mathcal{A} \) the set of such normalized holomorphic functions on the unit disk. And we denote by \( \mathcal{B} \) the set of normalized uniformly locally univalent functions: \( \mathcal{B} = \{ f \in \mathcal{A}; \|T_f\| < \infty \} \). The space \( \mathcal{B} \) has a structure of non-separable complex Banach space under the Hornich operation ([6]). Also this space is important in connection with the Teichmüller theory (cf. [1] and [9]). The amount of the norm \( \|T_f\| \) is thought to be strongly reflected by some geometric or analytic properties of the function \( f \), we will concern this quantity in the following.

For a non-negative real number \( \lambda \) we set
\[
\mathcal{B}(\lambda) = \{ f \in \mathcal{A}; \|T_f\| \leq 2\lambda \},
\]
here the number 2 is due to some technical reason.

In the class \( \mathcal{B}(\lambda) \) for \( 0 \leq \lambda < \infty \) the function
\[
F_\lambda(z) = \int_0^z \left( \frac{1+t}{1-t} \right)^\lambda \, dt
\]
is extremal as we shall see later. We note that \( F_\lambda \) is univalent if and only if \( 0 \leq \lambda \leq 1 \).
The following elementary fact is important for our argument below.

**Theorem 1** (Distortion Theorem). Let \( \lambda \) be a non-negative real number. For an \( f \in \mathcal{B}(\lambda) \) it holds that
\[
F'_\lambda(|z|) = \left( \frac{1-|z|}{1+|z|} \right)^\lambda \leq |f'(z)| \leq \left( \frac{1+|z|}{1-|z|} \right)^\lambda = F'_\lambda(|z|),
\]
and

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\[ |f(z)| \leq F_\lambda(|z|) \]

in the unit disk. Furthermore, if \( f \) is univalent then
\[ -F_\lambda(|z|) \leq |f(z)| \leq F_\lambda(|z|). \]

If the equality occurs in any of the above inequalities at some point \( z_0 \neq 0 \), then \( f \) must be a rotation of \( F_\lambda \), i.e., \( f(z) = \mu F_\lambda(\mu z) \) for a unimodular constant \( \mu \).

**Corollary 2.** For \( \lambda > 1 \) any \( f \in \mathcal{B}(\lambda) \) satisfies the growth condition
\[ f(z) = O\left(1 - |z|\right)^{1-\lambda} \]
as \( |z| \to 1 \). On the other hand, for \( \lambda < 1 \), a function \( f \in \mathcal{B}(\lambda) \) is always bounded with a uniform bound \( F_\lambda(1) \). Furthermore, if \( f \) is univalent, then \( f(\mathbb{D}) \) contains the disk \( \{|z| < -F_\lambda(-1)\} \). This constant \(-F_\lambda(-1)\) is best possible for \( 0 \leq \lambda \leq 1 \).

We note that for \( \lambda \leq 1/2 \) the function \( f \in \mathcal{B}(\lambda) \) must be univalent (cf. [2], [3]).

In this article, we will present several consequences from the above estimates and mention explicit norm estimates for various classes of univalent functions (for the proofs, see [4]). The following are sample theorems.

**Theorem 3.** Let \( 0 \leq \lambda < 1 \). Then any function \( f \in \mathcal{B}(\lambda) \) is Hölder continuous of exponent \( 1 - \lambda \) on the unit disk.

**Theorem 4.** Suppose \( f \in \mathcal{B}(\lambda) \) is univalent.
If \( \lambda < 1 \) then \( f \in H^\infty \).
If \( \lambda > 1 \) then \( f \in H^p \) for any \( 0 < p < 1/(\lambda - 1) \).
If \( \lambda = 1 \) then \( f \in \text{BMOA} \).

Note that \( H^\infty \subset \text{BMOA} \subset \cap_{0 < p < \infty} H^p \).
Let \( I_p(r, f) \) denote the integral mean of \( f \) with exponent \( p \in \mathbb{R} \):
\[ I_p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta. \]

And, for \( \lambda > 0 \), we set
\[ \alpha(\lambda) = \frac{\sqrt{1 + 4\lambda^2} - 1}{2}. \]

Note that
\[ \frac{\lambda^2}{\lambda + 1} < \alpha(\lambda) \leq \min\{\lambda^2, \frac{2\lambda^2}{2\lambda + 1}\} \leq \min\{\lambda^2, \lambda\}. \]

Then we have the following

**Theorem 5.** Let \( f(z) = z + a_2z^2 + a_3z^3 + \cdots \) be in \( \mathcal{B}(\lambda) \). Then, for any \( \varepsilon > 0 \) and a real number \( p \), we have \( I_p(r, f^\prime) = O\left(1 - r\right)^{-\alpha(p\lambda^{-1} - \varepsilon)} \); in particular, \( a_n = O(n^{\alpha(\lambda) - 1 + \varepsilon}) \).

Note that the extremal function \( F_\lambda \) has coefficients whose growth order is equivalent to \( r^{\lambda - 2} \).

The following is due to S. Yamashita. (The case of strongly starlike functions was first shown by [5].)

**Theorem A** (Yamashita [8]). Let \( 0 \leq \alpha < 1 \) and \( f \in S \).
If \( f \) is starlike of order \( \alpha \), i.e., \( \text{Re}(z f'(z)/f(z)) > \alpha \), then \( \|T_f\| \leq 6 - 4\alpha. \)
If $f$ is convex of order $\alpha$, i.e., $\text{Re}(1 +zf''(z)/f'(z)) > \alpha$, then $\|T_f\| \leq 4(1 - \alpha)$.

If $f$ is strongly starlike of order $\alpha$, i.e., $\text{arg}(zf'(z)/f(z)) < \pi\alpha/2$, then $\|T_f\| \leq M(\alpha) + 2\alpha$, where $M(\alpha)$ is a specified constant depending only on $\alpha$ satisfying $2\alpha < M(\alpha) < 2\alpha(1 + \alpha)$.

All of the bounds are sharp.

Finally we state general and useful principles for estimation of the norm of $T_f$. The following one always generates a sharp result for fixed $g$. The idea is due to Littlewood.

**Theorem 6** (Subordination Principle I). Let $g \in \mathcal{B}$ be given. For $f \in \mathcal{A}$, if $f'$ is subordinate to $g'$ then we have $\|T_f\| \leq \|T_g\|$. In particular, $f$ is uniformly locally univalent on the unit disk.

We can also show the next result.

**Theorem 7** (Subordination Principle II). Let $g \in \mathcal{B}$ be given. For $f \in \mathcal{A}$, if $zf'(z)/f(z)$ is subordinate to $g'$ then we have

$$
\|T_f\| \leq \sup_{z \in \mathbb{D}} (1 - |z|^p) \left( \left| \frac{g'(z)}{z} - 1 \right| + \|T_g(z)\| \right)
$$

$$
\leq \sup_{z \in \mathbb{D}} (1 - |z|^p) \left| \frac{g'(z)}{z} - 1 \right| + \|T_g\|.
$$

**References**


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