GROWTH AND COEFFICIENT ESTIMATES FOR UNIFORMLY LOCALLY UNIVALENT FUNCTIONS ON THE UNIT DISK

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The present article is a summary of our paper [4] which will appear somewhere.

We will call a holomorphic function f on the unit disk $\mathbb{D}$ *uniformly locally univalent* if f is univalent on each hyperbolic disk $D(a,\rho) = \{z \in \mathbb{D}; |\frac{z-a}{1-\bar{a}z}| < \tanh\rho\}$ with radius ρ and center $a \in \mathbb{D}$ for a positive constant ρ. It is well-known (cf. [7]) that a holomorphic function f on the unit disk is uniformly locally univalent if and only if the pre-Schwarzian derivative (or nonlinearity) $T_f = f''/f'$ of f is hyperbolically bounded, i.e., the norm

$$\|T_f\| = \sup_{z\in\mathbb{D}} (1-|z|^2)|T_f(z)|$$

is finite. This quantity can be regarded as the Bloch norm of the function $\log f'$.

Because T_f is invariant under the post-composition by a non-constant linear function, we may assume that a holomorphic function f on the unit disk is normalized so that $f(0) = 0$ and $f'(0) = 1$. We denote by $\mathcal{A}$ the set of such normalized holomorphic functions on the unit disk. And we denote by $\mathcal{B}$ the set of normalized uniformly locally univalent functions: $\mathcal{B} = \{f \in \mathcal{A}; \|T_f\| < \infty\}$. The space $\mathcal{B}$ has a structure of non-separable complex Banach space under the Hornich operation ([6]). Also this space is important in connection with the Teichmüller theory (cf. [1] and [9]). The amount of the norm $\|T_f\|$ is thought to be strongly reflected by some geometric or analytic properties of the function f, we will concern this quantity in the following.

For a non-negative real number λ we set

$$\mathcal{B}(\lambda) = \{f \in \mathcal{A}; \|T_f\| \le 2\lambda\},$$

here the number 2 is due to some technical reason.

In the class $\mathcal{B}(\lambda)$ for $0 \le \lambda < \infty$ the function

$$F_\lambda(z) = \int_0^z \left(\frac{1+t}{1-t}\right)^\lambda dt$$

is extremal as we shall see later. We note that F_λ is univalent if and only if $0 \le \lambda \le 1$. The following elementary fact is important for our argument below.

Theorem 1 (Distortion Theorem). *Let λ be a non-negative real number. For an $f \in \mathcal{B}(\lambda)$ it holds that*

$$F'_\lambda(-|z|) = \left(\frac{1-|z|}{1+|z|}\right)^\lambda \le |f'(z)| \le \left(\frac{1+|z|}{1-|z|}\right)^\lambda = F'_\lambda(|z|), \qquad \textit{and}$$

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$$|f(z)| \leq F_\lambda(|z|)$$

in the unit disk. Furthermore, if f is univalent then

$$-F_\lambda(-|z|) \leq |f(z)| \leq F_\lambda(|z|).$$

If the equality occurs in any of the above inequalities at some point $z_0 \neq 0$, then f must be a rotation of F_λ, i.e., $f(z) = \bar{\mu} F_\lambda(\mu z)$ for a unimodular constant μ.

Corollary 2. *For $\lambda > 1$ any $f \in \mathcal{B}(\lambda)$ satisfies the growth condition*

$$f(z) = O(1 - |z|)^{1-\lambda}$$

as $|z| \to 1$. On the other hand, for $\lambda < 1$, a function $f \in \mathcal{B}(\lambda)$ is always bounded with a uniform bound $F_\lambda(1)$. Furthermore, if f is univalent, then $f(\mathbb{D})$ contains the disk $\{|z| < -F_\lambda(-1)\}$. This constant $-F_\lambda(-1)$ is best possible for $0 \leq \lambda \leq 1$.

We note that for $\lambda \leq 1/2$ the function $f \in \mathcal{B}(\lambda)$ must be univalent (cf. [2], [3]).

In this article, we will present several consequences from the above estimates and mention explicit norm estimates for various classes of univalent functions (for the proofs, see [4]). The following are sample theorems.

Theorem 3. *Let $0 \leq \lambda < 1$. Then any function $f \in \mathcal{B}(\lambda)$ is Hölder continuous of exponent $1 - \lambda$ on the unit disk.*

Theorem 4. *Suppose $f \in \mathcal{B}(\lambda)$ is univalent.*
If $\lambda < 1$ then $f \in H^\infty$.
If $\lambda > 1$ then $f \in H^p$ for any $0 < p < 1/(\lambda - 1)$.
If $\lambda = 1$ then $f \in BMOA$.

Note that $H^\infty \subset BMOA \subset \cap_{0<p<\infty} H^p$.

Let $I_p(r, f)$ denote the integral mean of f with exponent $p \in \mathbb{R}$:

$$I_p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta.$$

And, for $\lambda > 0$, we set

$$\alpha(\lambda) = \frac{\sqrt{1 + 4\lambda^2} - 1}{2}.$$

Note that

$$\frac{\lambda^2}{\lambda + 1} < \alpha(\lambda) < \min\left\{\lambda^2, \frac{2\lambda^2}{2\lambda + 1}\right\} \leq \min\{\lambda^2, \lambda\}.$$

Then we have the following

Theorem 5. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ be in $\mathcal{B}(\lambda)$. Then, for any $\varepsilon > 0$ and a real number p, we have $I_p(r, f') = O(1 - r)^{-\alpha(|p|\lambda) - \varepsilon}$, in particular, $a_n = O(n^{\alpha(\lambda) - 1 + \varepsilon})$.*

Note that the extremal function F_λ has coefficients whose growth order is equivalent to $n^{\lambda - 2}$.

The following is due to S. Yamashita. (The case of strongly starlike functions was first shown by [5].)

Theorem A (Yamashita [8]). *Let $0 \leq \alpha < 1$ and $f \in \mathcal{S}$.*
If f is starlike of order α, i.e., $\mathrm{Re}(zf'(z)/f(z)) > \alpha$, then $\|T_f\| \leq 6 - 4\alpha$.

If f is convex of order α, i.e., $\mathrm{Re}(1 + zf''(z)/f'(z)) > \alpha$, *then* $\|T_f\| \le 4(1-\alpha)$.

If f is strongly starlike of order α, i.e., $\arg(zf'(z)/f(z)) < \pi\alpha/2$, *then* $\|T_f\| \le M(\alpha) + 2\alpha$, *where $M(\alpha)$ is a specified constant depending only on α satisfying* $2\alpha < M(\alpha) < 2\alpha(1+\alpha)$.

All of the bounds are sharp.

Finally we state general and useful principles for estimation of the norm of T_f. The following one always generates a sharp result for fixed g. The idea is due to Littlewood.

Theorem 6 (Subordination Principle I). *Let $g \in \mathcal{B}$ be given. For $f \in \mathcal{A}$, if f' is subordinate to g' then we have $\|T_f\| \le \|T_g\|$. In particular, f is uniformly locally univalent on the unit disk.*

We can also show the next result.

Theorem 7 (Subordination Principle II). *Let $g \in \mathcal{B}$ be given. For $f \in \mathcal{A}$, if $zf'(z)/f(z)$ is subordinate to g' then we have*

$$\begin{aligned}\|T_f\| &\le \sup_{z\in\mathbb{D}}(1-|z|^2)\left(\left|\frac{g'(z)-1}{z}\right| + |T_g(z)|\right)\\ &\le \sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{g'(z)-1}{z}\right| + \|T_g\|.\end{aligned}$$

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