ON THE BERS CONJECTURE FOR FUCHSIAN GROUPS OF THE SECOND KIND

DEDICATED TO PROFESSOR TATSUO FUJI'I'E ON HIS SIXTIETH BIRTHDAY

Toshiyuki Sugawa

§**1. Introduction.** Suppose that D is a simply connected domain of hyperbolic type in the Riemann sphere $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. Then the Poincaré metric ρ_D in D is defined by

$$\rho_D(z) = \frac{|g'(z)|}{1-|g(z)|^2}, \quad z \in D,$$

where g is any conformal mapping of D onto the unit disk $\Delta = \{z : |z| < 1\}$. $B_2(D)$ will denote the Banach space consisting of all holomorphic functions φ in D such that the norm

$$\|\varphi\|_D = \sup_{z\in D} |\varphi(z)|\rho_D(z)^{-2}$$

is finite.

If f is a locally univalent meromorphic function, the Schwarzian derivative of f is given by

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

We set after Flinn [6]

$$\begin{aligned} S &= \{S_f : f \text{ is conformal in } \Delta\}, \\ J &= \{S_f \in S \ : f(\Delta) \text{ is a Jordan domain }\}, \\ T &= \{S_f \in S \ : f(\Delta) \text{ is a quasidisk }\}. \end{aligned}$$

T is called the universal Teichmüller space. It is known that $T \subset J \subset S \subset B_2(\Delta), T$ is open, S is closed and $T = \mathrm{Int} S$ (see [7], [9]). Let $\mathit{\Gamma}$ be a Fuchsian group acting on Δ and $B_2(\Delta, \mathit{\Gamma})$ denote the closed subspace of $B_2(\Delta)$:

$$\{\varphi \in B_2(\Delta) : (\varphi\circ\gamma)\cdot(\gamma')^2 = \varphi \quad \text{for all} \quad \gamma \in \mathit{\Gamma}\}.$$

Further we set $S(\mathit{\Gamma}) = S \cap B_2(\Delta, \mathit{\Gamma}), J(\mathit{\Gamma}) = J \cap B_2(\Delta, \mathit{\Gamma}), T(\mathit{\Gamma}) = T \cap B_2(\Delta, \mathit{\Gamma})$. Then $T(\mathit{\Gamma})$ coincides with the Bers embedding of the Teichmüller space of $\mathit{\Gamma}$ (see [3]).

Bers conjectured that $S = \overline{T}$ i.e., $S(1) = \overline{T(1)}$ in [2]. Generalizing this conjecture, by *the Bers conjecture for* $\mathit{\Gamma}$ we shall mean that $S(\mathit{\Gamma}) = \overline{T(\mathit{\Gamma})}$.

In [8], Gehring showed that the Bers conjecture is false i.e., $S \setminus \overline{T} \neq \phi$, in fact essentially he showed that $S \setminus \overline{J} \neq \phi$. Moreover Flinn proved that $J \setminus \overline{T} \neq \phi$ in [6] (see the next section for the details). Our main purpose in this paper is to show the following

Theorem 1. *Suppose that Γ is a Fuchsian group of the second kind. Then $S(\Gamma) \setminus \overline{J} \neq \phi$ and $J(\Gamma) \setminus \overline{T} \neq \phi$.*

Corollary. *When Γ is of the second kind, the Bers conjecture for Γ is false. In other words, $S(\Gamma) \supsetneq \overline{T(\Gamma)}$.*

On the other hand, the author does not know whether the Bers conjecture for Fuchsian groups of the first kind is true or not.

This paper is organized as follows. In §2 we introduce simply connected domains which are constructed by Gehring and Flinn and quote two results from Flinn [6] for later use. Gehring's (or Flinn's) domain is essentially obtained by removing a spiral (respectively, countable sequence of closed Jordan regions approximating a spiral) from a disk so that the boundary of the domain is adequately irregular. For a given Fuchsian group Γ of the second kind, in §3 we construct a Γ-invariant simply connected domain Δ' in Δ which contains the Gehring's or Flinn's domain adjacent to a free side of the Dirichlet fundamental region of Γ. Let $F : \Delta \to \Delta'$ be a Riemann mapping function of Δ', then $S_F \in S(G)$ where $G = F^{-1}\Gamma F$ is a Fuchsian group. While it turns out later that $S_F \notin \overline{J}$ (respectively, $S_F \notin \overline{T}$) and that G is qc(=quasiconformally) equivalent to Γ (Lemma 4, Corollary), these facts need not imply that $S(\Gamma) \setminus \overline{J} \neq \phi$ (respectively, $J(\Gamma) \setminus \overline{T} \neq \phi$). Now we consider to deform Δ' by an appropriate qc mapping so that the Schwarzian derivative as above belongs to $S(\Gamma)$. In §4, such a deformation is presented and we state a slightly general result (Theorem 2) which contains Theorem 1. §5 is devoted to the proof of Theorem 2.

§2. Gehring's and Flinn's construction. Fix $a \in (0, \frac{1}{8\pi})$ and set a closed Jordan arc

$$\gamma_a = \{\pm i e^{(-a+i)t} : t \in [0, \infty)\} \cup \{0\}.$$

Theorem A (Gehring [8]). *Let $F : \Delta \to \widehat{\mathbf{C}} \setminus \gamma_a$ be a Riemann mapping function of $\widehat{\mathbf{C}} \setminus \gamma_a$, then $S_F \in S \setminus \overline{J}$.*

We set

$$A = \{x + iy : y > 1\} \cup \{x + iy : x > -4, y > -1\}$$

and $D_1 = A \setminus \gamma_a$. As we shall find later, the following theorem also holds.

Theorem A'. *Let $F : \Delta \to D_1$ be a Riemann mapping function of D_1, then $S_F \in S \setminus \overline{J}$.*

For each positive integer $m \in \mathbf{N}$, we set $\sigma_m = (\frac{\pi}{8})^m$, $\tau_m = e^{-2\pi a m}$, $E_m = R_m \cup P_m$ where

$$P_m = \{e^{i\sigma} z : z \in \gamma_a, -\sigma_m \leq \sigma \leq \sigma_m\} \cup \{z : |z| \leq \tau_m\},$$
$$R_m = \{x + iy : |x| \leq \sin\sigma_m, -1 \leq y \leq -\cos\sigma_m\} \setminus \Delta.$$

Then each E_m is a closed Jordan region, $E_1 \supset E_2 \supset \cdots$ and $\bigcap_{m=1}^{\infty} E_m = \gamma_a$. Let V denote the translation $V(z) = z + 8$ and set $D_2 = A \setminus \bigcup_{m=1}^{\infty} V^m(E_m)$. One can easily see that D_2 is a Jordan domain in $\widehat{\mathbf{C}}$. The next theorem is essentially due to Flinn.

Theorem B (Flinn [6, Theorem 2]**).** *Let $F : \Delta \to D_2$ be a Riemann mapping function of D_2 , then $S_F \in J \setminus \overline{T}$.*

Theorems A' and B are direct conclusions of the following Lemmas 1 and 2, respectively. Let $\alpha_1 = \{z = e^{(-a+i)t} : t \in (0,\infty)\}, \alpha_2 = \{z : -z \in \alpha_1\}$. Then α_1 and α_2 are logarithmic spirals in D_2 which converge onto the point 0 from opposite sides of γ_a.

Lemma 1 (cf. **Flinn** [6, Lemma 2]**).** *There exists a constant $\delta_1 > 0$ with the following property. If f is conformal in D_1 with $\|S_f\|_{D_1} \le \delta_1$, then*

$$\lim_{\alpha_1 \ni z \to 0} f(z) = \lim_{\alpha_2 \ni z \to 0} f(z).$$

In particular, $f(D_1)$ is not a Jordan domain.

Let β be the subarc $\{x + iy \in \partial D_1 : -4 < x < \infty\}$ of ∂D_1. We note that if $f : R_1 \to R_2$ is a conformal mapping of a Jordan domain R_1 onto another Jordan domain R_2, then f is uniquely extended to a homeomorphism $\tilde{f} : \overline{R_1} \to \overline{R_2}$.

Lemma 2 (cf. **Flinn** [6, Proof of Theorem 2]**).** *There exists a constant $\delta_2 > 0$ with the following property. If f is conformal in D_2 with $\|S_f\|_{D_2} \le \delta_2$ and if $f(D_2)$ is a Jordan domain, then $\tilde{f}(\beta)$ is not a quasiarc.*

Remark. In Thorems A' and B, we can replace the domain A by a half plane $\{x + iy : y > -1\}$.

§**3. Construction of group invariant domains.** Let Γ be an arbitrary Fuchsian group of the second kind acting on the unit disk Δ. In this section we construct Γ-invariant simply connected domains which have the same property as the Gehring's or Flinn's domain. Since Γ is of the second kind, $\Omega(\Gamma) \cap \partial\Delta \neq \phi$ where $\Omega(\Gamma)$ is the region of discontinuity of Γ in $\widehat{\mathbf{C}}$. Now we pick a sufficiently small disk Y in $\Omega(\Gamma)$ whose boundary is orthogonal to $\partial\Delta$ so that no two distinct points of Y are Γ-equivalent. Let σ be a Möbius transformation such that $\sigma(Y) = \Delta$ and that $\sigma(Y^+) = \Delta^+$ where $Y^+ = Y \cap \Delta$ and $\Delta^+ = \{z \in \Delta : \mathrm{Im} z > 0\}$. Fix $r_0, r_1 \in (0,1)$ such that $r_1 < r_0$. Let $\Delta_r = \{z \in \Delta : |z| < r\}$, $\Delta_r^+ = \Delta_r \cap \Delta^+$ and $Y_r^+ = \sigma^{-1}(\Delta_r^+)$ for $r \in (0,1)$.

Let $M(\Delta)$ be the space of Beltrami coefficients supported in $\Delta : \{\mu \in L^\infty(\Delta) : \|\mu\|_\infty < 1\}$. For $\mu \in M(\Delta), w^\mu$ will denote the normalized μ-conformal self-mapping of Δ which fixes three points $1, i, -1 \in \partial\Delta$. We set

$$M_{Y_{r_0}^+}(\Delta)_k = \{\mu \in L^\infty(\Delta) : \mu = 0 \text{ on } Y_{r_0}^+ \text{ and } \|\mu\|_\infty < k\},$$

for $k \in (0,1]$. Notice that w^μ is conformal in $Y_{r_0}^+$ for $\mu \in M_{Y_{r_0}^+}(\Delta)_k$.

Lemma 3. *Let $\delta_0 = \min\{\delta_1, \delta_2\}$ where δ_1 and δ_2 are as in Lemmas 1 and 2, respectively, then there exists a constant $k \in (0,1]$ such that for any $\mu \in M_{Y_{r_0}^+}(\Delta)_k$ the following is valid:*

$$\|S_{w^\mu}\|_{Y_{r_1}^+} \le \frac{\delta_0}{2}.$$

Proof. We can extend w^μ to a qc homeomorphism of $\widehat{\mathbf{C}}$ by the rule

$$w^\mu(z) = 1/\overline{w^\mu(1/\bar{z})}.$$

It is well known that $w^\mu(z)$ converges to z uniformly on each compact set in $\mathbf{C}$ as $\|\mu\|_\infty \to 0$ (see [1], for example). Since w^μ is conformal in $Y_{r_0} = \sigma^{-1}(\Delta_{r_0})$ for $\mu \in M_{Y^+_{r_0}}(\Delta)_1$ and $Y^+_{r_1}$ is relatively compact in Y_{r_0}, S_{w^μ} converges to 0 uniformly on $Y^+_{r_1}$ as $\|\mu\|_\infty \to 0$, in particular $\|S_{w^\mu}\|_{Y^+_{r_1}}$ converges to 0 as $\|\mu\|_\infty \to 0$. $\square$

For each $\varepsilon \in (0, r_1)$ we choose a $\tau = \tau_\varepsilon \in$Möb such that $\tau(\widehat{\mathbf{R}}) = \{x - i : x \in \mathbf{R}\} \cup \{\infty\}$ and $\tau(\Delta^+_\varepsilon) \supset A$.

Construction 1 (Gehring type).
We set $\Delta'_1 = \Delta'_1(\varepsilon) = \Delta \setminus \bigcup_{\gamma \in \Gamma} \gamma((\tau \circ \sigma)^{-1}(\gamma_a))$.

Construction 2 (Flinn type).
We set $\Delta'_2 = \Delta'_2(\varepsilon) = \Delta \setminus \bigcup_{\gamma \in \Gamma} \gamma((\tau \circ \sigma)^{-1}(\bigcup_{m=1}^\infty V^m(E_m)))$.

We note that $(\tau\circ\sigma)^{-1}(\gamma_a) \subset Y^+$, that $(\tau\circ\sigma)^{-1}(\bigcup_{m=1}^\infty V^m(E_m)) \subset Y^+$ and that $(\gamma(Y^+))_{\gamma\in\Gamma}$ is a disjoint family. Therefore Δ'_j is a Γ-invariant simply connected domain contained in Δ for $j = 1, 2$; furthermore Δ'_2 is a Jordan domain. If we let $F_j : \Delta \to \Delta'_j$ be a Riemann mapping function and set $G_j = F_j^{-1}\Gamma F_j$ which is a subgroup of Möb acting discontinuously on Δ i.e., a Fuchsian group acting on Δ, then $S_{F_1} \in S(G_1)$ and $S_{F_2} \in J(G_2)$.

Now we state a lemma which guarantees that G_j is qc equivalent to Γ.

Lemma 4. *Let $k \in (0, 1]$ be as in Lemma 3. For sufficiently small $\varepsilon \in (0, r_1)$, there exists a qc mapping h_j of $\Delta'_j = \Delta'_j(\varepsilon)$ onto Δ with the following properties for $j = 1, 2$.*

(1) $\|\mu(h_j)\|_\infty < k$ *where $\mu(h_j)$ is the Beltrami coefficient of h_j,*
(2) h_j *is conformal in* $\Delta'_j \cap Y^+_{r_0}$,
(3) $h_j \circ \gamma = \gamma \circ h_j$ *for all $\gamma \in \Gamma$.*

Because the qc mapping $f_j = h_j \circ F_j : \Delta \to \Delta$ deforms G_j into Γ, we have the following

Corollary. *G_j is qc equivalent to Γ.*

Lemma 4 is obtained in an obvious way by the following

Lemma 5. *For sufficiently small $\varepsilon \in (0, r_1)$, there exists a qc mapping H_j : $\sigma(Y'_j) \to \Delta^+$ with the following properties for $j = 1, 2$.*

(1) $\|\mu(H_j)\|_\infty < k$ *where $\mu(H_j)$ is the Beltrami coefficient of H_j,*
(2) H_j *is conformal in* $\sigma(Y'_j) \cap \Delta^+_{r_0}$,
(3) $H_j =$ identity *on* $\partial\Delta^+ \setminus [-1, 1]$,

where $Y'_j = Y'_j(\varepsilon) = Y \cap \Delta'_j(\varepsilon)$.

Proof of Lemma 5. Let $H_{(\varepsilon)} : \sigma(Y'_j(\varepsilon)) \cap \Delta^+_{r_0} \to \Delta^+_{r_0}$ be the conformal mapping which fixes three points $r_0, r_0 i, -r_0$. Noting that $\Delta^+_{r_0} \setminus \overline{\Delta^+_\varepsilon} \subset \sigma(Y'_j(\varepsilon))$, we find that

$H_{(\varepsilon)}|_{\Delta_{r_0}^+ \setminus \overline{\Delta_\varepsilon^+}}$ can be extended to a conformal mapping $\widetilde{H_{(\varepsilon)}}$ in $\{z : \varepsilon < |z| < {r_0}^2/\varepsilon\}$ by the reflection principle. Let $\Theta_\varepsilon : (0, \pi) \to (0, \pi)$ be the mapping defined by the rule $r_0 e^{i\Theta_\varepsilon(\theta)} = \widetilde{H_{(\varepsilon)}}(r_0 e^{i\theta})$ for all $\theta \in (0, \pi)$, then Θ_ε is a smooth mapping such that

$$\Theta_\varepsilon'(\theta) = |H_{(\varepsilon)}'(r_0 e^{i\theta})|.$$

Now we set

$$H_{(\varepsilon)}(re^{i\theta}) = re^{i(t\theta + (1-t)\Theta_\varepsilon(\theta))} \quad \text{for } r \in [r_0, 1], \theta \in (0, \pi),$$

where $r = t + (1-t)r_0$, then this extended $H_{(\varepsilon)}$ has the properties (2) and (3). On the other hand, clearly $\widetilde{H_{(\varepsilon)}}(z)$ converges to z uniformly on each compact set in $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ as $\varepsilon \to 0$, hence $\Theta_\varepsilon(\theta), \Theta_\varepsilon'(\theta)$ uniformly converges to $\theta, 1$ as $\varepsilon \to 0$, respectively. Therefore the explicit expression

$$|\mu(H_{(\varepsilon)})(re^{i\theta})| = \begin{cases} \left|\frac{(r-r_0)(\Theta_\varepsilon'(\theta)-1)-(\Theta_\varepsilon(\theta)-\theta)ri}{2(1-r_0)+(r-r_0)(\Theta_\varepsilon'(\theta)-1)+(\Theta_\varepsilon(\theta)-\theta)ri}\right|, & r \in (r_0, 1], \\ 0, & r \in (0, r_0) \end{cases}$$

shows that $\|\mu(H_{(\varepsilon)})\|_\infty \to 0$ as $\varepsilon \to 0$, so for sufficiently small $\varepsilon \in (0, r_1)$, $H_j = H_{(\varepsilon)} : \sigma(Y_j') \to \Delta^+$ has the properties (1), (2) and (3). $\square$

Henceforth we fix an $\varepsilon \in (0, r_1)$ for which Lemma 4 holds.

§**4. Deformations by partially conformal qc mappings.** In this section we present a method to construct the family of group-invariant domains which includes desired one.

Let $M(\Delta, \Gamma)$ be the space of Beltrami coefficients for Γ with support in Δ i.e., the subset of $L^\infty(\Delta)$ consisting of all $\mu \in L^\infty(\Delta)$ with $\|\mu\|_\infty < 1$ and

$$(\mu \circ \gamma) \cdot \overline{\gamma'}/\gamma' = \mu \quad \text{for all} \quad \gamma \in \Gamma.$$

Set $\mathcal{D}_j^\mu = w^\mu(\Delta_j')$ for $\mu \in M(\Delta, \Gamma)$. If Γ^μ denotes the Fuchsian group $w^\mu \Gamma (w^\mu)^{-1}$ acting on Δ, then $\mathcal{D}_j^\mu$ is a Γ^μ-invariant simply connected domain whose boundary is homeomophic to the Δ_j''s. We take a Riemann mapping function $F_j^\mu : \Delta \to \mathcal{D}_j^\mu$ and set $G_j^\mu = (F_j^\mu)^{-1} \Gamma^\mu F_j^\mu$ and $\varphi_j^\mu = S_{F_j^\mu}$. Since Γ^μ acts discontinuously on $\mathcal{D}_j^\mu$, G_j^μ acts also discontinuously on Δ, hence G_j^μ is a Fuchsian group. And clearly $\varphi_1^\mu \in S(G_1^\mu)$ and $\varphi_2^\mu \in J(G_1^\mu)$. Because the logarithmic spiral γ_a is a quasiarc, the general qc mapping w^μ may unfasten the spirals removed, thus we must restrict Beltrami coefficients μ to be considered on a certain class of $M(\Delta, \Gamma)$. In this article we only consider

$$\begin{aligned} M_{Y_{r_0}^+}(\Delta, \Gamma)_k &= M_{Y_{r_0}^+}(\Delta)_k \cap M(\Delta, \Gamma) \\ &= \{\mu \in M(\Delta, \Gamma) : \mu = 0 \text{ on } Y_{r_0}^+ \text{ and } \|\mu\|_\infty < k\}. \end{aligned}$$

Since w^μ is conformal in $Y_{r_0}^+$ for $\mu \in M_{Y_{r_0}^+}(\Delta, \Gamma)_k$, it is expected that the spirals are but slightly deformed by w^μ. In fact, we have the following result for this class which is proved in the rest of this paper.

Theorem 2. *Let k be as in Lemma 3. For $\mu \in M_{Y^+_{r_0}}(\Delta, \Gamma)_k$, we have*

$$\varphi_1^\mu \in S(G_1^\mu) \setminus \overline{J},$$

$$\varphi_2^\mu \in J(G_2^\mu) \setminus \overline{T}.$$

Theorem 2 and Lemma 4 prove Theorem 1.

Proof of Theorem 1. Let h_j be as in Lemma 4. We set

$$\mu_j = \begin{cases} \mu(h_j), & \text{on } \Delta'_j\,, \\ 0, & \text{on } \Delta \setminus \Delta'_j\,, \end{cases}$$

where $\mu(h_j)$ is the Beltrami coefficient of h_j and set $\mathcal{D}'_j = w^{\mu_j}(\Delta'_j)$. Notice that $\mu_j \in M_{Y^+_{r_0}}(\Delta, \Gamma)_k$. Since h_j and $w^{\mu_j}|_{\Delta'_j}$ have the same Beltrami coefficient, $w^{\mu_j} \circ h_j^{-1} : \Delta \to \mathcal{D}'_j$ is a Riemann mapping function of $\mathcal{D}'_j$. Therefore we can take $w^{\mu_j} \circ h_j^{-1}$ as $F_j^{\mu_j}$. By definition, $G_j^{\mu_j} = h_j \circ (w^{\mu_j})^{-1} \Gamma^{\mu_j} w^{\mu_j} \circ h_j^{-1} = h_j \Gamma h_j^{-1} = \Gamma$. By virture of Theorem 2, $\varphi_1^{\mu_1} \in S(\Gamma) \setminus \overline{J}, \varphi_2^{\mu_2} \in J(\Gamma) \setminus \overline{T}$, hence Theorem 1 is proved. $\square$

Remark. The family $(\varphi_j^\mu)_{\mu \in M_{Y^+_{r_0}}(\Delta, \Gamma)_k}$ is ample in a certain sense. We now explain this in the following. We recall that $F_j : \Delta \to \Delta'_j$ is a Riemann mapping function of Δ'_j and $G_j = F_j^{-1} \Gamma F_j$ is a Fuchsian group acting on Δ. In this paragraph we assume that Γ is non-elementary and choose F_j so that $1, i, -1 \in \Lambda(G_j) = \widehat{\mathbf{C}} \setminus \Omega(G_j)$. Let ${F_j}^* : M(\Delta, \Gamma) \to M(\Delta, G_j)$ be the pullback of Beltrami coefficients by F_j, namely ${F_j}^*(\mu)$ is the Beltrami coefficient of the qc mapping $w^\mu \circ F_j$ for $\mu \in M(\Delta, \Gamma)$. Since $w^{{F_j}^*(\mu)}$ and $w^\mu \circ F_j$ have the same Beltrami coefficient, we can choose $w^\mu \circ F_j \circ (w^{{F_j}^*(\mu)})^{-1} : \Delta \to \mathcal{D}_j^\mu$ as the Riemann mapping function F_j^μ, then we have $G_j^\mu = w^{{F_j}^*(\mu)} G_j (w^{{F_j}^*(\mu)})^{-1}$.

Generally, for $\nu \in M(\Delta, G_j)$ the group isomorphism $g \mapsto w^\nu g (w^\nu)^{-1}\ (g \in G_j)$ determines an element of the reduced Teichmüller space $T^\#(G_j)$ of G_j (see, for example, Earle [4], [5], Nag [10]). Let this point in $T^\#(G_j)$ be denoted by $\Phi^\#(\nu)$. It turns out that $\Phi^\#(M_K(\Delta, G_j)_k)$ is a neighborhood of $\Phi^\#(0)$ in $T^\#(G_j)$ for any $k \in (0, 1]$ and any measurable set $K \subset \Delta$ such that $p(K)$ is relatively compact in the double $\Omega(G_j)/G_j$ where $p : \Omega(G_j) \to \Omega(G_j)/G_j$ is the canonical projection. (For example, combine [11, Corollay 2] with [4]. This fact was pointed out to the author by H.Ohtake.)

On the other hand ${F_j}^*(M_{Y^+_{r_0}}(\Delta, \Gamma)_k) = M_{F_j^{-1}(Y^+_{r_0})}(\Delta, G_j)_k$ and $p(F_j^{-1}(Y_{r_0}))$ is relatively compact in $\Omega(G_j)/G_j$, hence $\Phi^\#({F_j}^*(M_{Y^+_{r_0}}(\Delta, \Gamma)_k))$ is a neighborhood of $\Phi^\#(0)$. In other words, qc deformations $G_j \to G_j^\mu\ (g \mapsto w^{{F_j}^*(\mu)} g (w^{{F_j}^*(\mu)})^{-1})$ for $\mu \in M_{Y^+_{r_0}}(\Delta, \Gamma)_k$ cover a neighborhood of the identity mapping $G_j \to G_j$ in $T^\#(G_j)$.

The above proof of Theorem 1 shows virtually that there exists an isomorphism $G_j \to \Gamma$ which belongs to $\Phi^\#({F_j}^*(M_{Y^+_{r_0}}(\Delta, \Gamma)_k))$.

§5. Proof of Theorem 2.

Proof of the first part. $\varphi_1^\mu \notin \overline{J}$.

By the same argument in [6] or [8], it is sufficient to prove the following

Claim 1. *There exists a constant $\delta > 0$ with the following property. If f is conformal in $\mathcal{D}_1^\mu$ with $\|S_f\|_{\mathcal{D}_1^\mu} \leq \delta$, then $f(\mathcal{D}_1^\mu)$ is not a Jordan domain.*

Proof of Claim 1. We set $\delta = \delta_1/2$ where δ_1 is as in Lemma 1. Suppose that f is conformal in $\mathcal{D}_1^\mu$ with $\|S_f\|_{\mathcal{D}_1^\mu} \leq \delta$. Set $g = f \circ w^\mu \circ (\tau \circ \sigma)^{-1}|_{D_1}$, $\beta_j = w^\mu \circ (\tau \circ \sigma)^{-1}(\alpha_j)$ for $j = 1, 2$ and $w_0 = w^\mu \circ (\tau \circ \sigma)^{-1}(0)$. Since

$$\|S_g\|_{D_1} = \|S_{f \circ w^\mu}\|_{(\tau \circ \sigma)^{-1}(D_1)} \leq \|S_f\|_{\mathcal{D}_1^\mu} + \|S_{w^\mu}\|_{Y_{r_1}^+} \leq \delta_1 \,,$$

Lemma 1 implies that

$$\lim_{\beta_1 \ni w \to w_0} f(w) = \lim_{\beta_2 \ni w \to w_0} f(w).$$

Thus $f(\mathcal{D}_1^\mu)$ is not a Jordan domain. □

Proof of the second part. $\varphi_2^\mu \notin \overline{T}$.

Similary it is sufficient to prove the following

Claim 2. *There exists a constant $\delta > 0$ with the following property. If f is conformal in $\mathcal{D}_2^\mu$ with $\|S_f\|_{\mathcal{D}_2^\mu} \leq \delta$, then $f(\mathcal{D}_2^\mu)$ is not a quasidisk.*

Proof of Claim 2. We set $\delta = \delta_2/2$ where δ_2 is as in Lemma 2. Suppose that f is conformal in $\mathcal{D}_2^\mu$ with $\|S_f\|_{\mathcal{D}_2^\mu} \leq \delta$. Further suppose that $f(\mathcal{D}_2^\mu)$ is a Jordan domain. We shall show that $\partial f(\mathcal{D}_2^\mu)$ is *not* a quasicircle. Set $g = f \circ w^\mu \circ (\tau \circ \sigma)^{-1}|_{D_2}$. Since $\|S_g\|_{D_2} \leq \|S_f\|_{\mathcal{D}_2^\mu} + \|S_{w^\mu}\|_{Y_{r_1}^+} \leq \delta_2$ and $g(D_2)$ is a Jordan domain, Lemma 2 produces that $\tilde{g}(\beta)$ is not a quasiarc. Hence $\partial f(\mathcal{D}_2^\mu)$ is not a quasicircle because $\tilde{g}(\beta) \subset \partial f(\mathcal{D}_2^\mu)$. □

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Department of Mathematics, Faculty of Science,
Kyoto University, Kyoto 606, Japan

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