A PROPERTY OF FUKUI'S EXTREMAL FUNCTION

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1. Introduction.

Let p be a positive integer and $\mathcal{A}_p$ denote the class of analytic functions f in the unit disk Δ with Taylor expansions of the form: $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$. For a constant $\alpha \in [0,1]$, a function f in $\mathcal{A}_p$ is called p-valently starlike of order α (respectively, p-valently convex of order α), if f satisfies the condition $\mathrm{Re}\frac{zf'}{f} \geq \alpha$ in Δ (respectively, $\mathrm{Re}(1+\frac{zf''}{f'}) \geq \alpha$ in Δ). We denote by $S_p^*(\alpha)$ and $K_p(\alpha)$ the class of p-valently starlike and convex functions of order α, respectively. The Marx-Strokhäcker theorem asserts that $K_1(0) \subset S_1^*(\frac{1}{2})$. Later, several authors have made efforts toward the generalization of this to the case for general p. At least, the following result has been proved by S. Fukui and M. Nunokawa.

Theorem 1.1. *For any $p \geq 2$, it holds that $K_p(0) \subset S_p^*(0)$.*

Recently, S. Fukui proved that there exists no positive constant $\alpha > 0$ such that $K_p(0) \subset S_p^*(\alpha)$. In fact, he introduced a function $f \in K_p(0)$ which is extremal in some sense, and satisfies that $\inf_{z\in\Delta} \mathrm{Re}\frac{zf'(z)}{f(z)} = 0$ (for a precise definition of f, see Section 2). Further, he exhibited that his function f can be written as $f(z) = \frac{z^p}{(z-1)^{2p}}h(z)$ with a polynomial h of degree p, and he showed a remarkable property that the real part of $h(e^{i\theta})$ is a constant multiple of $(1-\cos\theta)^p$ at least in case $p = 2, 3, 4, 5$.

In this note, we review Fukui's extremal function and give a complete (self-contained) proof for the next theorem, which is essentially equivalent to a classical result (Lemma 3.1) concerned with trigonometric series.

Theorem 1.2. *For any integer $p \geq 2$, Fukui's extremal function $f(z) = \frac{z^p}{(z-1)^{2p}}h(z)$ satisfies that*

$$\mathrm{Re}\, h(e^{i\theta}) = C_p(1-\cos\theta)^p,$$

where $C_p = 2^p\frac{(p!)^2}{(2p)!} = \frac{p(p-1)\cdots 2\cdot 1}{(2p-1)(2p-3)\cdots 3\cdot 1}$.

We should note that the above theorem was proved also by Yamakawa and Nunokawa, independently.

2. An extremality of Fukui's function.

For an integer $p \geq 2$, S. Fukui considered a function f in $\mathcal{A}_p$ which satisfies the differential equation:

$$f'(z) = \frac{pz^{p-1}}{(z-1)^{2p}}.$$

As is easily seen, the function f enjoies the property:

$$1 + \frac{zf''(z)}{f'(z)} = p \cdot \frac{1+z}{1-z}$$

in Δ, thus it follows that $f \in K_p(0)$. Here, we set $h(z) = \frac{(z-1)^{2p}}{z^p} f(z)$, then we have

$$\begin{aligned} h'(z) &= \frac{(z-1)^{2p}}{z^p} f'(z) + \{2pz - p(z-1)\} \frac{(z-1)^{2p-1}}{z^{p+1}} f(z) \\ &= \frac{p}{z} + \frac{p(z+1)}{z(z-1)} h(z). \end{aligned}$$

Thus, h is an analytic solution of the differential equation

$$zh'(z) + p \cdot \frac{1+z}{1-z} h(z) = p. \tag{2.1}$$

Let $h(z) = \sum_{n=0}^{\infty} A_n z^n$ be the power series expansion, then from (2.1) it follows that

$$\begin{aligned} &\sum_{n=0}^{\infty} nA_n z^n + p\left(1 + 2\sum_{n=1}^{\infty} z^n\right) \sum_{n=0}^{\infty} A_n z^n \\ &= \sum_{n=0}^{\infty} (n+p) A_n z^n + 2p \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} A_k\right) z^n = p, \end{aligned}$$

thus the coefficients should satisfy $A_0 = 1$ and

$$(n+p)A_n + 2p \sum_{k=0}^{n-1} A_k = 0 \quad (n \geq 1). \tag{2.2}$$

From (2.2) with $n = 1$, we see that $A_1 = -\frac{2p}{p+1}$. If $n \geq 2$, subtracting $(n+p-1)A_{n-1} + 2p \sum_{k=0}^{n-2} A_k = 0$ from (2.2), we have

$$(n+p)A_n + (p-n+1)A_{n-1} = 0.$$

Therefore, we obtain that $A_n = 0$ for $n > p$ and

$$\begin{aligned} A_n &= \left(-\frac{p-n+1}{p+n}\right)\left(-\frac{p-n+2}{p+n-1}\right)\cdots\left(-\frac{p-1}{p+2}\right)A_1 \\ &= 2(-1)^n\frac{p(p-1)\cdots(p-n+1)}{(p+n)\cdots(p+1)} \\ &= (-1)^n\frac{2(p!)^2}{(p+n)!(p-n)!} \end{aligned} \tag{2.3}$$

for $1 \le n \le p$. In particular, $h(z)$ is a polynomial of degree p. First note that $h(1) = 0$ by an equivalent relation of (2.1):

$$z(1-z)h'(z) + p(1+z)h(z) + p(z-1) = 0. \tag{2.4}$$

Differentiation of (2.4) yields that

$$z(1-z)h''(z) + (p+1+(p-2)z)h'(z) + ph(z) + p = 0, \tag{2.5}$$

in particular, $h'(1) = -\frac{p}{2p-1}$. Further differentiating (2.5), we obtain

$$z(1-z)h'''(z) + (p+2+(p-4)z)h''(z) + (2p-2)h'(z) = 0,$$

and $h''(1) + h'(1) = 0$, especially. Now we sum up the above result as

$$h(1) = 0, \quad h'(1) = -h''(1) = -\frac{p}{2p-1}(\neq 0).$$

Here we note that

$$\frac{zf'(z)}{f(z)} = \frac{z\cdot pz^{p-1}}{(z-1)^{2p}f(z)} = \frac{p}{h(z)},$$

so Theorem 1.1 implies that $\mathrm{Re}\frac{p}{h(z)} \ge 0$. Hence, we obtain that $\inf_{z\in\Delta}\mathrm{Re}\frac{p}{h(x)} = 0$ in fact by showing the following elementary

Lemma 2.1. *Suppose that $h(z)$ is analytic near $z = 1$ and that*

$$h(1) = 0, \quad h'(1) \neq 0.$$

Then we have

$$\liminf_{\Delta\ni z\to 1}\mathrm{Re}\frac{1}{h(z)} \le \frac{-\mathrm{Re}[h'(1)+h''(1)]}{2|h'(1)|^2}.$$

Proof. We write $h(e^{i\theta}) = u(\theta) + iv(\theta)$. Then we have

$$\begin{aligned} u'(\theta) + iv'(\theta) &= ie^{i\theta}h'(e^{i\theta}), \text{ and} \\ u''(\theta) + iv''(\theta) &= -e^{i\theta}h'(e^{i\theta}) - e^{2i\theta}h''(e^{i\theta}). \end{aligned}$$

Letting $\theta=0$, we have $u(0)=v(0)=0$,

$$\begin{aligned} &u'(0)^2+v'(0)^2=|h'(1)|^2, \qquad \text{and} \\ &u''(0)=-\mathrm{Re}[h'(1)+h''(1)]. \end{aligned}$$

Thus, by de l'Hospital's theorem, we see that

$$\begin{aligned} &\liminf_{\Delta\ni z\to 1}\mathrm{Re}\frac{1}{h(z)}=\liminf_{\Delta\ni z\to 1}\frac{\mathrm{Re}h(z)}{|h(z)|^2} \\ &\le \lim_{\theta\to 0}\frac{\mathrm{Re}h(e^{i\theta})}{|h(e^{i\theta})|^2}=\lim_{\theta\to 0}\frac{u(\theta)}{u(\theta)^2+v(\theta)^2}=\lim_{\theta\to 0}\frac{u'(\theta)}{2(u'(\theta)u(\theta)+v'(\theta)v(\theta))} \\ &=\frac{1}{2}\lim_{\theta\to 0}\frac{u''(\theta)}{u''(\theta)u(\theta)+u'(\theta)^2+v''(\theta)v(\theta)+v'(\theta)^2} \\ &=\frac{u''(0)}{2(u'(0)^2+v'(0)^2)}=\frac{-\mathrm{Re}[h'(1)+h''(1)]}{2|h'(1)|^2}. \end{aligned}$$

□

3. Proof of Theorem 1.2.

This section devoted to a proof of Theorem 1.2. We remark that this theorem produces a more direct proof of the extremality of Fukui's function: $\inf_{z\in\Delta}\mathrm{Re}\frac{p}{h(z)}=0$.

First, we prepare the next

Lemma 3.1. *For a positive integer p,*

$$\sin^{2p}\frac{\theta}{2}=\sum_{n=0}^{p}2^{1-\varepsilon-2p}(-1)^n\binom{2p}{p-n}\cos n\theta, \tag{3.1}$$

where $\varepsilon=\delta_{0,n}$, i.e., $\varepsilon=1$ if $n=0$ and $\varepsilon=0$ otherwise.

However this is a known result, we include a proof for convenience of the reader.

Proof. Since the function $\sin^{2p}\frac{\theta}{2}$ is even, its Fourier expansion takes a form: $\sin^{2p}\frac{\theta}{2}=\sum_{n=0}^{\infty}B_n\cos n\theta$, here

$$B_n=\frac{1}{2^{\varepsilon}\pi}\int_0^{2\pi}\sin^{2p}\frac{\theta}{2}\cos n\theta d\theta$$

for $n=0,1,2,\cdots$. Now we introduce an auxiliary double sequence

$$\alpha_{k,n}=\int_0^{2\pi}\sin^{2k}\frac{\theta}{2}\cos n\theta d\theta$$

for $k, n = 0, 1, 2, \cdots$. Then, in order to prove the assertion, it suffices to show that

$$\alpha_{k,n} = \begin{cases} \pi(-1)^n 2^{1-2k} \begin{pmatrix} 2k \\ k-n \end{pmatrix} & \text{if } 0 \leq n \leq k, \\ 0 & \text{if } k < n. \end{cases}$$

Using a descending formula:

$$\begin{aligned}
\alpha_{k,n} &= \int_0^{2\pi} \sin^{2k-1} \frac{\theta}{2} \left(\sin \frac{\theta}{2} \cos n\theta \right) d\theta \\
&= \frac{1}{2} \int_0^{2\pi} \sin^{2k-1} \frac{\theta}{2} \left(\sin(n + \frac{1}{2})\theta - \sin(n - \frac{1}{2})\theta \right) d\theta \\
&= \frac{1}{4} \int_0^{2\pi} \sin^{2k-2} \frac{\theta}{2} \Big[(\cos n\theta - \cos(n+1)\theta) - (\cos(n-1)\theta - \cos n\theta) \Big] d\theta \\
&= \frac{1}{4} (2\alpha_{k-1,n} - \alpha_{k-1,n+1} - \alpha_{k-1,n-1}),
\end{aligned}$$

we can show the above equation by induction with respect to k. $\square$

Now we return to Fukui's extremal function f. By (2.3), we have

$$\begin{aligned}
h(z) &= \sum_{n=0}^{p} A_n z^n = 1 + \sum_{n=1}^{p} (-1)^n \frac{2(p!)^2}{(p+n)!(p-n)!} z^n \\
&= \sum_{n=0}^{p} (-1)^n 2^{1-\varepsilon} \frac{(p!)^2}{(p+n)!(p-n)!} z^n.
\end{aligned}$$

In particular, we obtain

$$\text{Re} h(e^{i\theta}) = \sum_{n=0}^{p} (-1)^n 2^{1-\varepsilon} \frac{(p!)^2}{(p+n)!(p-n)!} \cos n\theta. \tag{3.2}$$

On the other hand, we have

$$(1 - \cos\theta)^p = \left(2 \sin^2 \frac{\theta}{2} \right)^p = \sum_{n=0}^{p} (-1)^n 2^{1-\varepsilon-p} \begin{pmatrix} 2p \\ p-n \end{pmatrix} \cos n\theta$$

by (3.1), and

$$C_p 2^{1-\varepsilon-p} \begin{pmatrix} 2p \\ p-n \end{pmatrix} = 2^{1-\varepsilon} \frac{(p!)^2}{(2p)!} \frac{(2p)!}{(p+n)!(p-n)!} = 2^{1-\varepsilon} \frac{(p!)^2}{(p+n)!(p-n)!},$$

hence now Theorem 1.2 is proved.