A PROPERTY OF FUKUI’S EXTREMAL FUNCTION

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1. Introduction.

Let \( p \) be a positive integer and \( \mathcal{A}_p \) denote the class of analytic functions \( f \) in the unit disk \( \Delta \) with Taylor expansions of the form:

\[
    f(z) = z^p + a_{p+1}z^{p+1} + \cdots.
\]

For a constant \( \alpha \in [0, 1] \), a function \( f \) in \( \mathcal{A}_p \) is called \( p \)-valently starlike of order \( \alpha \) (respectively, \( p \)-valently convex of order \( \alpha \)), if \( f \) satisfies the condition \( \text{Re} \frac{zf'}{f(z)} \geq \alpha \) in \( \Delta \) (respectively, \( \text{Re}(1 + \frac{zf'}{f(z)}) \geq \alpha \) in \( \Delta \)). We denote by \( S^*_p(\alpha) \) and \( K_p(\alpha) \) the class of \( p \)-valently starlike and convex functions of order \( \alpha \), respectively. The Marx-Strohhäcker theorem asserts that \( K_1(0) \subseteq S^*_1(\frac{1}{2}) \). Later, several authors have made efforts toward the generalization of this to the case for general \( p \). At least, the following result has been proved by S. Fukui and M. Nunokawa.

**Theorem 1.1.** For any \( p \geq 2 \), it holds that \( K_p(0) \subset S^*_p(0) \).

Recently, S. Fukui proved that there exists no positive constant \( \alpha > 0 \) such that \( K_p(0) \subset S^*_p(\alpha) \). In fact, he introduced a function \( f \in K_p(0) \) which is extremal in some sense, and satisfies that \( \inf_{z \in \Delta} \text{Re} \frac{zf'}{f(z)} = 0 \) (for a precise definition of \( f \), see Section 2). Further, he exhibited that his function \( f \) can be written as \( f(z) = \frac{z^p}{(z-1)^p} h(z) \) with a polynomial \( h \) of degree \( p \), and he showed a remarkable property that the real part of \( h(e^{i\theta}) \) is a constant multiple of \((1 - \cos \theta)^p\) at least in case \( p = 2, 3, 4, 5 \).

In this note, we review Fukui’s extremal function and give a complete (self-contained) proof for the next theorem, which is essentially equivalent to a classical result (Lemma 3.1) concerned with trigonometric series.

**Theorem 1.2.** For any integer \( p \geq 2 \), Fukui’s extremal function

\[
    f(z) = \frac{z^p}{(z-1)^p} h(z)
\]

satisfies that

\[
    \text{Re} \; h(e^{i\theta}) = C_p(1 - \cos \theta)^p,
\]

where

\[
    C_p = 2^p \frac{(p)!^2}{(2p)!} = \frac{p(p-1)\cdots 2 \cdot 1}{(2p-1)(2p-3)\cdots 3 \cdot 1}.
\]

We should note that the above theorem was proved also by Yamakawa and Nunokawa, independently.
2. An extremality of Fukui's function.

For an integer \( p \geq 2 \), S. Fukui considered a function \( f \) in \( A_p \) which satisfies the differential equation:

\[
f'(z) = \frac{pz^{p-1}}{(z-1)^{2p}}.
\]

As is easily seen, the function \( f \) enjoys the property:

\[
1 + \frac{zf''(z)}{f'(z)} = p \cdot \frac{1 + z}{1 - z}
\]

in \( \Delta \), thus it follows that \( f \in K_p(0) \). Here, we set \( h(z) = \frac{(z-1)^{2p}}{z^p} f(z) \), then we have

\[
h'(z) = \frac{(z-1)^{2p}}{z^p} f'(z) + \{2pz - p(z - 1)\} \frac{(z-1)^{2p-1}}{z^{p+1}} f(z)
\]

\[
= \frac{p}{z} + \frac{p(z + 1)}{z(z - 1)} h(z).
\]

Thus, \( h \) is an analytic solution of the differential equation

\[
zh'(z) + p \cdot \frac{1 + z}{1 - z} h(z) = p.
\]

Let \( h(z) = \sum_{n=0}^{\infty} A_n z^n \) be the power series expansion, then from (2.1) it follows that

\[
\sum_{n=0}^{\infty} nA_n z^n + p \left( 1 + 2 \sum_{n=1}^{\infty} z^n \right) \sum_{n=0}^{\infty} A_n z^n
\]

\[
= \sum_{n=0}^{\infty} (n + p) A_n z^n + 2p \sum_{n=1}^{\infty} \left( \sum_{k=0}^{n-1} A_k \right) z^n = p,
\]

thus the coefficients should satisfy \( A_0 = 1 \) and

\[
(2.2) \quad (n + p)A_n + 2p \sum_{k=0}^{n-1} A_k = 0 \quad (n \geq 1).
\]

From (2.2) with \( n = 1 \), we see that \( A_1 = -\frac{2p}{p+1} \). If \( n \geq 2 \), subtracting \( (n+p-1)A_{n-1} + 2p \sum_{k=0}^{n-2} A_k = 0 \) from (2.2), we have

\[
(n + p)A_n + (p - n + 1)A_{n-1} = 0.
\]

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Therefore, we obtain that \( A_n = 0 \) for \( n > p \) and
\[
A_n = \left( -\frac{p-n+1}{p+n} \right) \left( -\frac{p-n+2}{p+n-1} \right) \cdots \left( -\frac{p-1}{p+2} \right) A_1
\]
\[
= 2(-1)^n \frac{p(p-1) \cdots (p-n+1)}{(p+n) \cdots (p+1)}
\]
\[
= (-1)^n \frac{2(p!)^2}{(p+n)!(p-n)!}
\]
(2.3)

for \( 1 \leq n \leq p \). In particular, \( h(z) \) is a polynomial of degree \( p \). First note that \( h(1) = 0 \) by an equivalent relation of (2.1):
\[
z(1-z)h'(z) + p(1+z)h(z) + p(z-1) = 0.
\]
(2.4)

Differentiation of (2.4) yields that
\[
z(1-z)h''(z) + (p+1+(p-2)z)h'(z) + ph(z) + p = 0,
\]
in particular, \( h'(1) = -\frac{p}{2p-1} \). Further differentiating (2.5), we obtain
\[
z(1-z)h'''(z) + (p+2+(p-4)z)h''(z) + (2p-2)h'(z) = 0,
\]
and \( h''(1) + h'(1) = 0 \), especially. Now we sum up the above result as
\[
h(1) = 0, \quad h'(1) = -h''(1) = -\frac{p}{2p-1} (\neq 0).
\]

Here we note that
\[
\frac{zf'(z)}{f(z)} = \frac{z \cdot p \cdot z^{p-1}}{(z-1)^{2p} f(z)} = \frac{p}{h(z)},
\]
so Theorem 1.1 implies that \( \Re \frac{p}{h(z)} \geq 0 \). Hence, we obtain that \( \inf_{z \in \Delta} \Re \frac{p}{h(z)} = 0 \) in fact by showing the following elementary

**Lemma 2.1.** Suppose that \( h(z) \) is analytic near \( z = 1 \) and that
\[
h(1) = 0, \quad h'(1) \neq 0.
\]
Then we have
\[
\lim \inf_{\Delta \ni \xi} \Re \frac{1}{h(z)} \leq \frac{-\Re [h'(1) + h''(1) \sqrt{2|h'(1)|}]}{2|h'(1)|^2}.
\]

*Proof.* We write \( h(e^{i\theta}) = u(\theta) + iv(\theta) \). Then we have
\[
u'(\theta) + iv'(\theta) = ie^{i\theta}h'(e^{i\theta}), \quad \text{and}
\]
\[
u''(\theta) + iv''(\theta) = -e^{i\theta}h'(e^{i\theta}) - e^{2i\theta}h''(e^{i\theta}).
\]
Letting $\theta = 0$, we have $u(0) = v(0) = 0$,

$$u'(0)^2 + v'(0)^2 = |h'(1)|^2,$$

and

$$u''(0) = -\text{Re}[h'(1) + h''(1)].$$

Thus, by de l'Hospital’s theorem, we see that

$$\liminf_{A \Delta z \to 1} \frac{1}{h(z)} = \liminf_{A \Delta z \to 1} \frac{\text{Re} h(z)}{|h(z)|^2}$$

$$\leq \lim_{\theta \to 0} \frac{\text{Re}(e^{i\theta})}{|h(e^{i\theta})|^2} = \lim_{\theta \to 0} \frac{u(\theta)}{u(\theta)^2 + v(\theta)^2} = \lim_{\theta \to 0} \frac{u'(\theta)}{2(u'(\theta)v(\theta) + v'(\theta)v(\theta))}$$

$$= \frac{1}{2} \lim_{\theta \to 0} \frac{u''(\theta)v(\theta) + v''(\theta)v(\theta) + v'(\theta)^2}{u'(\theta)^2 + v'(\theta)^2}$$

$$= \frac{u''(0)}{2(u'(0)^2 + v'(0)^2)} = -\frac{\text{Re}[h'(1) + h''(1)]}{2|h'(1)|^2}.$$

□

3. Proof of Theorem 1.2.

This section devoted to a proof of Theorem 1.2. We remark that this theorem produces a more direct proof of the extremality of Fukui’s function: $\inf_{x \in A} \text{Re} \frac{F}{h(z)} = 0$.

First, we prepare the next

**Lemma 3.1.** For a positive integer $p$,

\begin{equation}
\sin^{2p} \frac{\theta}{2} = \sum_{n=0}^{p} 21^{n-2p}(-1)^n \left( \frac{2p}{n - p} \right) \cos n\theta,
\end{equation}

where $\varepsilon = \delta_{0,n}$, i.e., $\varepsilon = 1$ if $n = 0$ and $\varepsilon = 0$ otherwise.

However this is a known result, we include a proof for convenience of the reader.

**Proof.** Since the function $\sin^{2p} \frac{\theta}{2}$ is even, its Fourier expansion takes a form: $\sin^{2p} \frac{\theta}{2} = \sum_{n=0}^{\infty} B_n \cos n\theta$, here

$$B_n = \frac{1}{2\pi} \int_{0}^{2\pi} \sin^{2p} \frac{\theta}{2} \cos n\theta d\theta$$

for $n = 0, 1, 2, \ldots$. Now we introduce an auxiliary double sequence

$$a_{k,n} = \int_{0}^{2\pi} \sin^{2k} \frac{\theta}{2} \cos n\theta d\theta$$
for \( k, n = 0, 1, 2, \ldots \). Then, in order to prove the assertion, it suffices to show that

\[
\alpha_{k,n} = \begin{cases} 
\pi (-1)^n 2^{1-2k} \binom{2k}{k-n} & \text{if } 0 \leq n \leq k, \\
0 & \text{if } k < n.
\end{cases}
\]

Using a descending formula:

\[
\alpha_{k,n} = \int_0^{2\pi} \sin^{2k-1} \frac{\theta}{2} \left( \sin \frac{\theta}{2} \cos n\theta \right) d\theta
= \frac{1}{2} \int_0^{2\pi} \sin^{2k-1} \frac{\theta}{2} \left( \sin (n + \frac{1}{2})\theta - \sin (n - \frac{1}{2})\theta \right) d\theta
= \frac{1}{4} \int_0^{2\pi} \sin^{2k-2} \frac{\theta}{2} \left[ (\cos n\theta - \cos(n+1)\theta) - (\cos(n-1)\theta - \cos n\theta) \right] d\theta
= \frac{1}{4} (2\alpha_{k-1,n} - \alpha_{k-1,n+1} - \alpha_{k-1,n-1}),
\]

we can show the above equality by induction with respect to \( k \). \( \square \)

Now we return to Fukui’s extremal function \( f \). By (2.3), we have

\[
h(z) = \sum_{n=0}^{p} A_n z^n = 1 + \sum_{n=1}^{p} (-1)^n \frac{2(p!)^2}{(p+n)!(p-n)!} z^n
= \sum_{n=0}^{p} (-1)^n 2^{1-\epsilon} \frac{(p!)^2}{(p+n)!(p-n)!} z^n.
\]

In particular, we obtain

\[
(3.2) \quad \text{Re} h(e^{i\theta}) = \sum_{n=0}^{p} (-1)^n 2^{1-\epsilon} \frac{(p!)^2}{(p+n)!(p-n)!} \cos n\theta.
\]

On the other hand, we have

\[
(1 - \cos \theta)^p = \left( 2 \sin^2 \frac{\theta}{2} \right)^p = \sum_{n=0}^{p} (-1)^n 2^{1-\epsilon-p} \binom{2p}{p-n} \cos n\theta
\]

by (3.1), and

\[
C_{p} 2^{1-\epsilon-p} \left( \frac{2p}{p-n} \right) = 2^{1-\epsilon} \frac{(p!)^2}{(2p)!} \frac{(2p)!}{(p+n)!(p-n)!} = 2^{1-\epsilon} \frac{(p!)^2}{(p+n)!(p-n)!},
\]

hence now Theorem 1.2 is proved.