On uniform perfectness of limit sets of Kleinian groups and Julia sets of rational functions

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Abstract

In this note, we shall give a definition of uniformly perfect sets and then discuss the fundamental properties of these sets. As typical and important examples, we shall refer to the limit sets of Kleinian groups and the Julia sets of rational functions on the Riemann sphere.

A closed set E containing at least three points in the Riemann sphere $\widehat{\boldsymbol{C}}$ is said to be uniformly perfect if there exists a constant $0 < c < 1$ such that $E \cap \{z \in \boldsymbol{C}; cr < |z-a| < r\} \neq \emptyset$ for any $a \in E \setminus \{\infty\}$ and $0 < r < \mathrm{diam} E$. We set $D = \widehat{\boldsymbol{C}} \setminus E$. This is equivalent to any of the following.

1. $M_D = \sup_A m(A) < +\infty$, where A runs over all ring domains in D separating E and $m(A)$ denotes the modulus of A, i.e., A is conformally equivalent to $\{z \in \boldsymbol{C}; 1 < |z| < e^{m(A)}\}$.

2. $M_D^\circ = \sup_A m(A) < +\infty$, where A runs over all round annuli in D separating E and the round annulus means a ring domain of the form $\{r_1 < |z-a| < r_2\}$.

3. $\inf_\alpha \ell_D(\alpha) > 0$, where the infimum is taken over all non-trivial closed curves in D and $\ell_D(\alpha)$ denotes the hyperbolic length of the curve α, where the hyperbolic length means the length measured by the hyperbolic metric on D of constant negative curvature -4.

Remark that the domain constants M_D and L_D can be defined for (a disjoint union of) arbitrary hyperbolic Riemann surfaces. And further we set $L_D^* = \inf_\alpha \ell_D(\alpha)$, where the infimum is taken over all non-trivial closed curves in D not homotopic to any puncture. Of course, $L_D \geq L_D^*$ and if $L_D^* > 0$ we will say that D is of Lehner type. It is a standard fact that the space of integrable holomorphic quadratic differentials on D is contained in the space of (hyperbolically) bounded ones if and only if D is of Lehner type.

It is known that a uniformly perfect set is of positive Hausdorff dimension ([2]). More quantitatively, it holds that (cf. [7]),

$$\text{H-dim} E \geq \frac{\log 2}{\log(2e^{M_D^\circ}+1)} \geq \frac{\log 2}{M_D^\circ + \log 3}.$$

Pommerenke [4] proved that the limit set of a finitely generated non-elementary Kleinian group is uniformly perfect. We can generalize this result as follows. Let G be a non-elementary Kleinian group with the region of discontinuity $\Omega(G)$. For simplicity, we assume that G is torsion-free. (Actually, we shall state the corresponding result in the general case. For a precise statement, see [6].)

Theorem 1 *If the quotient surface $\Omega(G)/G$ is of Lehner type, then the limit set $\Lambda(G) = \widehat{\boldsymbol{C}} \setminus \Omega(G)$ is uniformly perfect.*

In fact, we can prove that $L(\Omega(G)) \geq L^*(\Omega(G)/G)$.

As for the Julia set of a rational function of degree ≥ 2, first Pommerenke [4] showed that it is uniformly perfect in the hyperbolic case. Afterwards, Mañé-da Rocha [3] and Hinkkanen [1] proved independently the uniform perfectness of the Julia sets in general case. But, their proof were done by contradiction, so no explicit bounds are given. In this note we can provide a geometric lower bound for the constant L_Ω, where Ω is the Fatou set of a rational function of degree ≥ 2.

Let f be a rational function of degree $d \geq 2$. Then f can be regarded as a holomorphic map from the Riemann sphere $\widehat{\boldsymbol{C}}$ onto itself. Let $\mathrm{Crit}(f)$ be the set of critical points of f in Ω. We note that $\#\mathrm{Crit}(f) \leq 2d-2$. Put

$$C_1 = \min_{v_1 \neq v_2 \in f(\mathrm{Crit}(f))} 2d_\Omega(v_1, v_2), \quad C_2 = \min_{v \in f(\mathrm{Crit}(f))} 4\iota_\Omega(v),$$

where d_Ω and ι_Ω denote the hyperbolic distance and the injectivity radius in Ω, respectively. It is important that $C_1 > 0$ and $C_2 > 0$.

Let $A_1, \cdots, A_t$ be the complete system of representatives of the cycles of Herman rings of f. We note here that, by Shishikura's theorem, $0 \leq t \leq d-2$, in particular, if $d = 2$ there are no Herman rings. Put $C_3 = \min\{L_{A_1}, \cdots, L_{A_t}\}$, then $C_3 > 0$ because the Julia set has no isolated points.

Under the above notation, we can state our result in the following form.

Theorem 2 *For an arbitrary rational function $f : \widehat{\boldsymbol{C}} \to \widehat{\boldsymbol{C}}$ of degree $d \geq 2$, the following holds.*

$$L_{\Omega_f} \geq \min\{C_1, C_2, C_3\}.$$

Actually, the above theorem follows from a somewhat stronger result. For details and the proof, see [5].

References

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