ON POSITIVITY OF TAYLOR COEFFICIENTS OF CONFORMAL MAPS

TOSHIYUKI SUGAWA

Abstract. We provide an approach to the proof of positivity of the Taylor coefficients for a given conformal map of the unit disk onto a plane domain. This short note is a summary of the joint work [2] with Stanisława Kanas.

1. Introduction

If a univalent function $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ has non-negative Taylor coefficients about the origin, namely, $a_k \ge 0$ for all $k \ge 0$, various sharp estimates can easily be deduced. For example, one can show the sharp inequalities

$$|f(z) - a_0 - a_1 z - \cdots - a_k z^k| \le f(|z|) - a_0 - a_1|z| - \cdots - a_k|z|^k$$

and

$$|f^{(k)}(z)| \le f^{(k)}(|z|)$$

for $k = 0, 1, 2, \ldots$. Note that this sort of inequalities are, in general, not easy to establish.

As one immediately sees, a necessary condition for a univalent function f to have non-negative Taylor coefficients is that the image domain $\Omega = f(\mathbb{D})$ is symmetric in the real axis. Under the assumption of this symmetric property, however, it seems to be difficult to give a sufficient condition for non-negativity of the coefficients in terms of the shape of Ω. For instance, the convexity of Ω is not sufficient. In fact, for a constant $0 < c < 1$, the function

$$f(z) = \frac{z}{1 + cz} = z - cz^2 + c^2 z^3 - c^3 z^4 + \cdots$$

maps $\mathbb{D}$ univalently onto a disk but has a negative coefficient. (In general, when $f(z)$ has non-negative Taylor coefficients, the function $\hat{f}(z) = -f(-z)$ has a negative coefficient unless f is an odd function.)

In this note, we will explain one approach to show positivity of the Taylor coefficients of a specific conformal map of the interior of a conic section.

2. Conformal mappings onto domains bounded by conic sections

For $k \in [0, \infty)$, we set

$$\Omega_k = \{u + iv \in \mathbb{C}; u^2 > k^2(u-1)^2 + k^2 v^2, u > 0\}.$$

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Note that $1 \in \Omega_k$ for all k. Ω_0 is nothing but the right half plane. When $0 < k < 1$, Ω_k is the unbounded domain enclosed by the right half of the hyperbola

$$\left(\frac{u + k^2/(1-k^2)}{k/(1-k^2)}\right)^2 - \frac{v^2}{1/(1-k^2)} = 1$$

with focus at 1. Ω_1 becomes the unbounded domain enclosed by the parabola

$$v^2 = 2u - 1$$

with focus at 1. When $k > 1$, the domain Ω_k is the interior of the ellipse

$$\left(\frac{u - k^2/(k^2-1)}{k/(k^2-1)}\right)^2 + \frac{v^2}{1/(k^2-1)} = 1$$

with focus at 1. For every k, the domain Ω_k is convex and symmetric in the real axis. Note also that $\Omega_{k_1} \supset \Omega_{k_2}$ if $0 \le k_1 \le k_2$.

Kanas and Wiśniowska [3] treated the family Ω_k in their study of k-uniformly convex functions and gave the explicit formulae for the conformal homeomorphisms $p_k : \mathbb{D} \to \Omega_k$ determined by $p_k(0) = 1$ and $p_k'(0) > 0$. Here, an analytic function $f(z)$ in the unit disk with $f(0) = 0, f'(0) = 1$ is called k-uniformly convex if the function $1 + zf''(z)/f'(z)$ maps the unit disk analytically into Ω_k. A function is 1-uniformly convex precisely when it is uniformly convex (see [4]).

In order to state their result, we prepare some notation. Let $\mathcal{K}(z,t)$ and $\mathcal{K}(t)$ be the normal and complete elliptic integrals, respectively, i.e.,

$$\mathcal{K}(z,t) = \int_0^z \frac{dx}{\sqrt{(1-x^2)(1-t^2x^2)}}$$

and $\mathcal{K}(t) = \mathcal{K}(1,t)$. The quantity

$$\mu(t) = \frac{\pi \mathcal{K}(\sqrt{1-t^2})}{2\mathcal{K}(t)}$$

is known as the modulus of the Groetszch ring $\mathbb{D} \setminus [0,t]$ for $0 < t < 1$. Note that $\mu(t)$ is a strictly decreasing smooth function. For details, see [1].

Proposition 1 (Kanas-Wiśniowska [3])**.** *The conformal map $p_k : \mathbb{D} \to \Omega_k$ with $p_k(0) = 1$ and $p_k'(0) > 0$ is given by*

$$p_k(z) = \begin{cases} (1+z)/(1-z) & \text{if } k = 0, \\ (1-k^2)^{-1}\cosh[C_k \log(1+\sqrt{z})/(1-\sqrt{z})] - k^2/(1-k^2) & \text{if } 0 < k < 1, \\ 1 + (2/\pi^2)[\log(1+\sqrt{z})/(1-\sqrt{z})]^2 & \text{if } k = 1, \\ (k^2-1)^{-1}\sin[C_k \mathcal{K}((z/\sqrt{t}-1)/(1-\sqrt{t}z), t)] + k^2/(k^2-1) & \text{if } 1 < k, \end{cases}$$

where $C_k = (2/\pi)\arccos k$ for $0 < k < 1$ and $C_k = \pi/2\mathcal{K}(t)$ and $t \in (0,1)$ is chosen so that $k = \cosh(\mu(t)/2)$ for $k > 1$.

3. Main Results

For each $k \in [0, \infty)$, we write

$$p_k(z) = 1 + A_1(k)z + A_2(k)z^2 + \cdots$$

for the conformal mapping p_k of $\mathbb{D}$ onto Ω_k with $p_k(0) = 1$ and $p_k'(0) > 0$. Since Ω_k lies in the right half-plane, Carathéodory's theorem yields that $|A_n(k)| \leq 2$ holds for each $n \geq 1$ and $k \in [0, \infty)$. Our main result is the following.

Theorem 2. *$A_n(k) > 0$ for all $n \geq 1$ and $k \in [0, +\infty)$.*

Since $p_0(z) = 1 + 2z + 2z^2 + 2z^3 + \cdots$ and

$$p_1(z) = 1 + \frac{2}{\pi^2}\left(z + \frac{z^2}{3} + \frac{z^3}{5} + \cdots\right)^2,$$

the assertion of the theorem is trivial for $k = 0$ and $k = 1$. When $0 < k < 1$, the assertion is also trivial because the function cosh has the non-negative Taylor coefficients.

In what follows, we consider the cases when $k > 1$. Due to complexity of the representation of p_k given above for $k > 1$, we try to simplify it.

We now consider the conformal mapping J of $\mathbb{D}$ onto $\widehat{\mathbb{C}} \setminus [-1, 1]$ defined by $f(z) = (z + z^{-1})/2$. Since

$$J(e^{-s+it}) = \cosh s \cos t - i \sinh s \sin t,$$

the circle $|z| = e^{-s}$ is mapped by J onto the ellipse E_s given by

$$\left(\frac{u}{\cosh s}\right)^2 + \left(\frac{v}{\sinh s}\right)^2 = 1$$

for $s > 0$ and the radial segment $(0, e^{it})$ is mapped by J into the component H_t of the hyperbola given by

$$\left(\frac{u}{\cos t}\right)^2 - \left(\frac{v}{\sin t}\right)^2 = 1, \quad u \cos t > 0,$$

for $t \in \mathbb{R}$ with $(2/\pi)t \notin \mathbb{Z}$.

Let T_n be the Chebyshev polynomial of degree n, i.e., $T_n(\cos\theta) = \cos(n\theta)$. Then it is well known that the n-fold mapping $z \mapsto z^n$ is conjugate under J to T_n, in other words,

$$J(z^n) = T_n(J(z))$$

holds in $|z| < 1$. In particular, one can see that the ellipse E_s is mapped by T_n onto E_{ns} and that the hyperbola H_t is mapped by T_n onto H_{nt}.

Applying the above argument to $T_2(w) = 2w^2 - 1$, we obtain the following.

Lemma 3. *The Chebyshev polynomial $T_2(w) = 2w^2 - 1$ maps the domain bounded by H_t and $H_{\pi-t}$ onto the connected component of $\mathbb{C} \setminus H_{2t}$ containing -1. Also, T_2 maps the domain bounded by the ellipse E_s onto the domain bounded by E_{2s}.*

On the basis of the above lemma, we can obtain another representation of p_k.

Theorem 4. *For $k > 0$, the function p_k is written by $p_k(z) = 1 + Q_k(\sqrt{z})^2$, where*

$$Q_k(z) = \begin{cases} \sqrt{\frac{2}{1-k^2}}\sinh(C_k \operatorname{arctanh} z) & \text{if} \quad 0 < k < 1, \\ \sqrt{\frac{1}{2\pi^2}}\operatorname{arctanh} z & \text{if} \quad k = 1, \\ \sqrt{\frac{2}{k^2-1}}\sin\left(C_k'\mathcal{K}(z/\sqrt{s}, s)\right) & \text{if} \quad 1 < k. \end{cases}$$

Here, $C_k = (2/\pi)\arccos k$ when $0 < k < 1$, and $s \in (0, 1)$ is chosen so that $k = \cosh\mu(s)$ and $C_k' = (\pi/2)/\mathcal{K}(s)$ when $k > 1$.

Furthermore, the function Q_k is odd and maps the unit disk conformally onto the domain $D_k = \{x + iy : (k-1)x^2 + (k+1)y^2 < 1\}$.

Note that D_k is the inside of a hyperbola when $k < 1$ and D_k is the interior of an ellipse when $k > 1$. When $k = 1$, the domain D_k becomes the parallel strip $-1/\sqrt{2} < \operatorname{Im} z < 1/\sqrt{2}$. Also note that D_k is invariant under the involution $z \mapsto -z$.

4. Rough idea of the proof

We indicate here how to deduce Theorem 2. A detailed exposition will appear in [2].

In order to prove positivity of the Taylor coefficients of p_k, it is enough to show that of Q_k thanks to Theorem 4. Though the assertion is trivial in the case when $0 < k < 1$, we first treat this case in order to highlight an idea of the present method. When $0 < k < 1$, one can check that $w = Q_k(z)$ satisfies the linear differential equation

$$(1 - z^2)^2 w'' - 2z(1 - z^2)w' - C_k^2 w = 0 \tag{1}$$

in $\mathbb{D}$.

Lemma 5. *Let $Q(z)$ be an analytic solution of (1) in $\mathbb{D}$ with $Q(0) = 0$ and $Q'(0) > 0$. Then Q has Taylor expansion in the form $Q(z) = \sum_{n=0}^{\infty} B_n z^{2n+1}$ and the coefficients satisfy the inequalities*

$$(2n + 1)B_n - (2n - 1)B_{n-1} > 0 \quad \textit{and} \quad B_n > 0 \tag{2}$$

for each $n \geq 1$.

Proof. By the linear differential equation (1), one obtains the recursive formula for coefficients

$$(2n + 2)(2n + 3)B_{n+1} - \left\{2(2n + 1)^2 + C_k^2\right\}B_n + 2n(2n - 1)B_{n-1} = 0$$

for $n \geq 0$, here we have set $B_{-1} = 0$. We now suppose that the assertion is true up to n. Then, by the above formula, we get

$$\begin{aligned} &(2n + 2)\left\{(2n + 3)B_{n+1} - (2n + 1)B_n\right\} \\ =&\left\{2(2n + 1)^2 - (2n + 2)(2n + 1) + C_k^2\right\}B_n - 2n(2n - 1)B_{n-1} \\ \geq&\left\{2(2n + 1)^2 - (2n + 2)(2n + 1)\right\}B_n - 2n(2n - 1)B_{n-1} \\ =&2n\left\{(2n + 1)B_n - (2n - 1)B_{n-1}\right\} > 0 \end{aligned} \tag{3}$$

Therefore, the assertion is also true for $n + 1$. By induction, the proof is done. ∎

In the case when $k > 1$, the function $w = Q_k(z)$ satisfies the similar differential equation

$$(1 - sz^2)(1 - z^2/s)w'' - 2z((s + s^{-1})/2 - z^2)w' + \frac{{C'_k}^2}{s}w = 0$$

in $\mathbb{D}$, where $s \in (0, 1)$ is chosen so that $k = \cosh \mu(s)$ and $C'_k = \pi/2\mathcal{K}(s)$. Note that $Q_k(z)$ satisfies $Q_k(0) = 0$ and $Q'_k(0) > 0$.

The above two differential equations can also be unified into the form

$$(1 - 2Mz^2 + z^4)w'' - 2z(M - z^2)w' - cw = 0, \tag{4}$$

where $M = 1$ and $c = C_k^2$ for $0 < k < 1$ and $M = (s + s^{-1})/2 \geq 1$ and $c = -{C'_k}^2/s = -\pi^2/4s\mathcal{K}(s)^2$ for $k > 1$. Let $w = Q(z)$ be the solution of the equation with the initial condition $Q(0) = 0$ and $Q'(0) = 1$. In the same way as above, one obtains the relations for the coefficients of $Q(z) = \sum_{n=0}^{\infty} B_n z^{2n+1}$:

$$(2n + 2)(2n + 3)B_{n+1} - \left\{2M(2n + 1)^2 + c\right\}B_n + 2n(2n - 1)B_{n-1} = 0 \tag{5}$$

for $n \geq 0$, where we also have set $B_{-1} = 0$.

In the case when $k > 1$, however, the above argument breaks down at the inequality (3) because now $c < 0$. In fact, the coefficients B_n tend rapidly to 0 as $n \to \infty$, therefore, some renormalization techniques are required in this case. See [2] for the details.

References

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Department of Mathematics, Graduate School of Science, Hiroshima University, Higashi-Hiroshima, 839-8526 Japan

E-mail address: `sugawa@math.sci.hiroshima-u.ac.jp`