COMPLETELY MONOTONE SEQUENCES AND UNIVERSALLY PRESTARLIKE FUNCTIONS

STEPHAN RUSCHEWEYH, LUIS SALINAS, AND TOSHIYUKI SUGAWA

Abstract. We introduce universally convex, starlike and prestarlike functions in the slit domain $\mathbb{C} \setminus [1, \infty)$, and show that there exists a very close link to completely monotone sequences and Pick functions.

1. Introduction

1.1. **Completely monotone sequences.** A sequence $\{a_k\}_{k\geq 0}$ of non-negative real numbers, $a_0 = 1$, is called *completely monotone* [1] (c.m.) if

$$\Delta^n a_k \quad := \quad \Delta^{n-1} a_k - \Delta^{n-1} a_{k+1} \, \geq \, 0, \quad k \geq 0, \; n \geq 1,$$

where Δ^0 is the identity operator: $\Delta^0 a = a$. It is a well-known result of Hausdorff [6] that $\{a_k\}_{k\geq 0}$ is c.m. if and only if there is a probability measure[2] μ on $[0, 1]$ such that

$$a_k = \int_0^1 t^k \, d\mu(t), \quad k \geq 0,$$

or, equivalently,

$$F(z) = \sum_{k=0}^{\infty} a_k \, z^k \; = \int_0^1 \frac{d\mu(t)}{1 - tz}.$$

Let $\mathcal{T}$ denote the set of such functions F. They are analytic in the slit domain $\Lambda := \mathbb{C} \setminus [1, \infty)$ and also belong to the set of Pick functions $P(-\infty, 1)$ (see Donoghue [5] for a detailed account of Pick functions). The study of c.m. sequences has a long history and they are of great importance in various fields of mathematics and statistics. It is the aim of this paper to exhibit a strong connection between these sequences and a class of functions well-known in geometric function theory. This leads, on the sequences side, to the construction of apparently new operators preserving c.m. sequences and to the explicit construction of c.m. sequences.

Date: March 9, 2007.

2000 *Mathematics Subject Classification.* Primary 30C45, 30E05, Secondary 44A60, 33C05.

Key words and phrases. Universally prestarlike functions, universally convex functions, completely monotone sequences, totally monotone sequences, Pick functions, hypergeometric functions, polylogarithms.

S.R. and L.S. received partial support from FONDECYT (grants 1070269 and 7070131) and DGIP-UTFSM (grant 240421). S.R. acknowledges partial support also from the German-Israeli Foundation (grant G-809-234.6/2003) and a JSPS Grant-in-Aid for Scientific Research (A), 17204010. T.S. was partially supported by JSPS Grant-in-Aid for Scientific Research (B), 17340039.

[1]Many authors use *totally monotone* instead of *completely monotone*.

[2]Measures in this paper are always understood to be Borel

1.2. **Prestarlike functions.** Let $\mathcal{H}(\Omega)$ denote the set of analytic functions in a domain Ω. For domains Ω containing the origin $\mathcal{H}_0(\Omega)$ stands for the set of functions $f \in \mathcal{H}(\Omega)$ with $f(0) = 1$. We also use the notation $\mathcal{H}_1(\Omega) := \{zf : f \in \mathcal{H}_0(\Omega)\}$. In the special case that Ω is the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ we use the abbreviations $\mathcal{H}, \mathcal{H}_0, \mathcal{H}_1$, respectively.

A function $f \in \mathcal{H}_1$ is called *starlike of order* α (with $\alpha < 1$) if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq \alpha, \quad z \in \mathbb{D},$$

and the set of such functions is denoted by $\mathcal{S}_\alpha$. Then, finally, a function $f \in \mathcal{H}_1$ is called *prestarlike of order* α if

$$\frac{z}{(1-z)^{2-2\alpha}} * f(z) \ \in \ \mathcal{S}_\alpha, \tag{1.1}$$

where '$*$' stands for the Hadamard product of two functions in $\mathcal{H}$:

$$g(z) = \sum_{k=0}^{\infty} g_k z^k, \ h(z) = \sum_{k=0}^{\infty} h_k z^k \ \Rightarrow g * h(z) := \sum_{k=0}^{\infty} g_k h_k z^k.$$

The sets of these functions are denoted by $\mathcal{R}_\alpha$. For certain reasons one also introduces the set $\mathcal{R}_1$ to consist of the functions $f \in \mathcal{H}_1$ with

$$\operatorname{Re} \frac{f(z)}{z} \geq \frac{1}{2}, \quad z \in \mathbb{D}.$$

Prestarlike functions have a number of interesting geometric properties. For instance, the set $\mathcal{C}$ of univalent functions in $\mathcal{H}_1$ which map $\mathbb{D}$ onto convex domains equals $\mathcal{R}_0$, and obviously we also have $\mathcal{R}_{1/2} = \mathcal{S}_{1/2}$. We refer to Ruscheweyh [15] and Sheil-Small [18] for a description of the essentials of the theory of prestarlike functions. For the present paper the most relevant information is the following result.

Lemma 1.1. *For $\alpha < \beta \leq 1$ we have $\mathcal{R}_\alpha \subset \mathcal{R}_\beta$.*

While working with prestarlike functions and convolutions the following notation turned out to be useful:

$$(D^\beta f)(z) := \frac{z}{(1-z)^\beta} * f, \quad \beta \geq 0.$$

In particular, for $\beta = n \in \mathbb{N}$ we have

$$D^{n+1} f = \frac{z}{n!} (z^{n-1} f)^{(n)}.$$

Using this operator we find that a function $f \in \mathcal{H}_1$ is prestarlike of order $\alpha \leq 1$ if and only if

$$\frac{D^{3-2\alpha} f}{D^{2-2\alpha} f} \in \mathcal{P}. \tag{1.2}$$

Here

$$\mathcal{P} = \{f \in \mathcal{H}_0 : \operatorname{Re} f(z) > \frac{1}{2}, \ z \in \mathbb{D}\},$$

or, equivalently, by the Herglotz formula,

$$f \in \mathcal{P} \ \Leftrightarrow f(z) = \int_0^{2\pi} \frac{d\mu(t)}{1 - e^{-it} z}, \tag{1.3}$$

where μ is a probability measure on $[0, 2\pi]$. One should observe that for any function $f \in \mathcal{T}$ the restriction of f to $\mathbb{D}$ is in $\mathcal{P}$.

1.3. **Universally prestarlike functions.** The basic aim of this paper, as far as geometric function theory is concerned, is a translation of the notion of prestarlike functions from the unit disc to other discs and half-planes containing the origin. Let Ω be one such disc or half-plane. Then there are two unique parameters $\gamma \in \mathbb{C} \setminus \{0\}$ and $\rho \in [0,1]$ such that

$$\Omega = \{w_{\gamma,\rho}(z) \, : \, z \in \mathbb{D}\}$$

where

$$w_{\gamma,\rho}(z) := \frac{\gamma z}{1-\rho z}.$$

To make this relation visible, we also write $\Omega_{\gamma,\rho}$ for this Ω.

Definition 1.1. Let $\alpha \leq 1$ and $\Omega = \Omega_{\gamma,\rho}$ for some admissible pair (γ, ρ). A function $f \in \mathcal{H}_1(\Omega)$ is called *prestarlike of order α in Ω* if

$$f_{\gamma,\rho}(z) := \frac{1}{\gamma} f(w_{\gamma,\rho}(z)) \in \mathcal{R}_\alpha.$$

In this paper we are not dealing with such functions specifically, but with functions being prestarlike of a given order in several sets $\Omega_{\gamma,\rho}$ simultaneously. Of course, it makes no sense to ask for functions which are prestarlike in *all* such sets, because then we are left with only the identity function. The situation changes already dramatically, if we admit exactly those $\Omega_{\gamma,\rho}$ which omit one given point, for instance the point 1. Note that $1 \notin \Omega_{\gamma,\rho}$ if and only if $|\gamma + \rho| \leq 1$

Definition 1.2. Let $\alpha \leq 1$. A function $f \in \mathcal{H}_1(\Lambda)$ is called *universally prestarlike of order α* if and only if f is prestarlike of order α in all sets $\Omega_{\gamma,\rho}$ with $|\gamma+\rho| \leq 1$. The set of these functions is denoted by $\mathcal{R}_\alpha^{\mathrm{u}}$.

Recall the definition of the set Λ and the family $\mathcal{T}$ from Section 1.1. Clearly, we have $\mathcal{R}_\alpha^{\mathrm{u}} \subset \mathcal{H}_1(\Lambda)$ and the main result of this paper is

Theorem 1.1. *Let $\alpha \leq 1$ and $f \in \mathcal{H}_1(\Lambda)$. Then $f \in \mathcal{R}_\alpha^{\mathrm{u}}$ if and only if*

$$\frac{D^{3-2\alpha} f}{D^{2-2\alpha} f} \in \mathcal{T}. \tag{1.4}$$

This admits an explicit representation of the functions in $\mathcal{R}_\alpha^{\mathrm{u}}$. If $f \in \mathcal{H}_0$ has all its Taylor coefficients at the origin different from zero we write $\{f\}^{(-1)}$ for the (possibly formal but) unique solution of $f * \{f\}^{(-1)} = \frac{1}{1-z}$. Note that for $\gamma > 0$ we have

$$I_\gamma(z) := \left\{ \frac{1}{(1-z)^\gamma} \right\}^{(-1)} = {}_2F_1(1,1,\gamma,z),$$

where ${}_2F_1$ stands for the hypergeometric function (see [1]).

Corollary 1.1. *$f \in \mathcal{H}_1(\Lambda)$ is universally prestarlike of order $\alpha < 1$ if and only if there exists a probability measure μ on $[0,1]$ such that*

$$\frac{f(z)}{z} = I_{2-2\alpha}(z) * \exp\left(\int_0^1 \log \frac{1}{(1-tz)^{2-2\alpha}} d\mu(t) \right).$$

Note that Lemma 1.1 can be stated as: for $f \in \mathcal{H}_1$ we have

$$\frac{D^{3-2\alpha}f}{D^{2-2\alpha}f} \in \mathcal{P} \Rightarrow \frac{D^{3-2\beta}f}{D^{2-2\beta}f} \in \mathcal{P} \Rightarrow \frac{f(z)}{z} \in \mathcal{P}, \quad \alpha < \beta < 1.$$

A combination of this with Theorem 1.1 now gives an equivalent result.

Corollary 1.2. *For $\alpha < \beta \le 1$ we have $\mathcal{R}^{\mathrm{u}}_\alpha \subset \mathcal{R}^{\mathrm{u}}_\beta$. In particular, for $f \in \mathcal{H}_1(\Lambda)$,*

$$\frac{D^{3-2\alpha}f}{D^{2-2\alpha}f} \in \mathcal{T} \Rightarrow \frac{D^{3-2\beta}f}{D^{2-2\beta}f} \in \mathcal{T} \Rightarrow \frac{f(z)}{z} \in \mathcal{T}, \quad \alpha < \beta < 1.$$

To make this property a little more transparent we mention that Lemma 1.1 contains an old result of Strohhäcker, namely that for $f \in \mathcal{H}_1$ we have

$$1 + \frac{zf''(z)}{2f'(z)} \in \mathcal{P} \;\Rightarrow\; \frac{zf'(z)}{f(z)} \in \mathcal{P} \;\Rightarrow\; \frac{f(z)}{z} \in \mathcal{P},$$

and that Corollary 1.2 implies that for $f \in \mathcal{H}_1(\Lambda)$ we have similarly

$$1 + \frac{zf''(z)}{2f'(z)} \in \mathcal{T} \;\Rightarrow\; \frac{zf'(z)}{f(z)} \in \mathcal{T} \;\Rightarrow\; \frac{f(z)}{z} \in \mathcal{T}.$$

Clearly Corollary 1.2 can be looked at as the preservation of the c.m. property of sequences under certain complicated operations. As a very simple example of this kind of interpretation we mention the following result.

Corollary 1.3. *Let $\sigma, \lambda_k > 0, t_k \in [0,1]$ for $1 \le k \le n$ with $\sum_{k=1}^n \lambda_k \le \sigma$. Then,*

$$I_\sigma(z) * \prod_{k=1}^n \frac{1}{(1-t_k z)^{\lambda_k}} \;\in\; \mathcal{T}.$$

1.4. **General properties of universally prestarlike functions.** The functions $f \in \widetilde{\mathcal{T}} := \{zF(z) : F \in \mathcal{T}\}$ have been studied for their geometric properties on several occasions, for instance by Wirths [22], and one of his general results was that the members in $\widetilde{\mathcal{T}}$ are univalent (in fact, convex in the direction of the real axis) in the half-plane $\{z : \operatorname{Re} z < 1\}$, but that there is no larger domain for which univalence generally holds. The universally prestarlike functions belong to $\widetilde{\mathcal{T}}$ by Theorem 1.1. Therefore this result applies to $\mathcal{R}^{\mathrm{u}}_\alpha$. For $\alpha \le \frac{1}{2}$ we can do even better.

Theorem 1.2. *Let f be universally prestarlike of order $\alpha \le 1$. Then f is univalent in the half-plane $\{z : \operatorname{Re} z < 1\}$. If $\alpha \le \frac{1}{2}$ then f is univalent in the whole of Λ and maps Λ onto a doman starlike w.r.t. the origin.*

It is easily seen that $f(z) = z$ is the only entire function in $\mathcal{R}^{\mathrm{u}}_\alpha$ for $\alpha \le 1$. However, $\mathcal{R}^{\mathrm{u}}_1$ has obviously many rational members. This changes for $\alpha \le \frac{1}{2}$.

Theorem 1.3. *The functions $f(z) := z/(1-tz),\ t \in [0,1]$ are the only rational functions in $\mathcal{R}^{\mathrm{u}}_{1/2}$.*

Theorem 1.4. *Let $\alpha \le 1$ and $f \in \mathcal{R}^{\mathrm{u}}_\alpha$. Then the functions*

$$-f\left(\frac{z}{z-1}\right), \quad \frac{1}{t}f(tz),\ t \in (0,1),$$

belong to $\mathcal{R}^{\mathrm{u}}_\alpha$ as well.

1.5. **Universally starlike and convex functions.** Historically the first general classes of conformal mappings in $\mathbb{D}$ studied in greater detail were those which map $\mathbb{D}$ onto starlike or convex domains. Therefore we include the 'universal' concept also for these classes.

Definition 1.3. A function $f \in \mathcal{H}_1(\Lambda)$ is called *universally starlike* (w.r.t. the origin) if $f_{\gamma,\rho}$ belongs to $\mathcal{S}_0$ in $\mathbb{D}$ for each pair (γ, ρ) with $|\gamma + \rho| \leq 1$.

Definition 1.4. A function $f \in \mathcal{H}_1(\Lambda)$ is called *universally convex* if it maps every half-plane containing the point 1 in its boundary and the origin in its interior univalently onto a convex domain.

One should notice the systematical difference in these definitions, mainly based on the fact that convexity is translation invariant, which is not the case for starlikeness w.r.t. the origin. The following somewhat surprising results give additional justification for studying universally prestarlike functions of order α.

Theorem 1.5. *A function f is universally starlike w.r.t. the origin if and only if it is universally prestarlike of order $\frac{1}{2}$.*

Theorem 1.6. *A function is universally convex if and only if it is universally prestarlike of order 0. Such f maps every disc and every half-plane contained in Λ onto a convex domain.*

Remark Since $\mathcal{R}_{1/2} = \mathcal{S}_{1/2}$, the above result shows that the restriction to $\mathbb{D}$ of a universally starlike function belongs to $\mathcal{S}_{1/2}$. A remarkable consequence of this is that the fundamental Alexander theorem, namely $f \in \mathcal{C} \Leftrightarrow zf'(z) \in \mathcal{S}_0$, is no longer valid for universally convex and universally starlike functions: $f(z) = \frac{z}{1-z}$ is universally convex but the restriction of $zf'(z)$ to $\mathbb{D}$ does not belong to $\mathcal{S}_{1/2}$, and this shows that the (famous) Koebe function $zf'(z) = \frac{z}{(1-z)^2}$ is actually not universally starlike.

1.6. **Examples.** In this section we present two general examples for universally starlike and convex functions.

1.6.1. *Polylogarithms.* J. Lewis [10], with an extremely involved proof, showed that the polylogarithmic functions

$$\mathrm{Li}_\alpha(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^\alpha} \qquad \alpha \geq 0,$$

are convex univalent in $\mathbb{D}$. Since, for $\alpha > 0$,

$$g_\alpha(z) := \frac{\mathrm{Li}_\alpha(z)}{z} = \frac{1}{\Gamma(\alpha)} \int_0^1 \frac{(\log(1/t))^{\alpha-1} dt}{1 - tz}, \tag{1.5}$$

we see that $g_\alpha \in \mathcal{T}$ for $\alpha > 0$ and hence Li_α belong to $\mathcal{R}_1^{\mathrm{u}}$ (clearly both statements hold for $\alpha = 0$ as well). Therefore the question arises whether these functions are also *universally* convex. We have the following result.

Theorem 1.7. *The functions Li_α are universally starlike for $\alpha = 0$ and $\alpha \geq 1$. They are universally convex for $\alpha = 0, \alpha = 1$ and $\alpha \geq 2$. For $\alpha \in (0, 1)$ they are not universally convex.*

For $a \in (0,1)$ we have

$$G_\alpha(z) := 1 + \frac{z}{2}\frac{\mathrm{Li}_\alpha''(z)}{\mathrm{Li}_\alpha'(z)} = 1 + \frac{1}{2^\alpha}\, z + \left(\frac{1}{3^{\alpha-1}} - \frac{1}{2^{2\alpha-1}}\right) z^2 + \dots .$$

It is easy to check that, the third coefficient in this expansion is larger than the second one. Therefore G_α cannot be in $\mathcal{T}$, and Theorem 1.1 then implies that Li_α cannot be universally convex for those α. However, the following extension of Theorem 1.7 seems possible.

Conjecture 1.1. *The functions* Li_α *are universally starlike for* $\alpha \geq 0$ *and universally convex for* $\alpha \geq 1$.

If this is true, then Li_α maps Λ univalently onto a domain starlike with respect to the origin for every $\alpha \geq 0$, which would nicely add to Lewis' previously mentioned result.

1.6.2. *Hypergeometric functions.* The hypergeometric functions ${}_2F_1(a,b,c,z)$ have been studied in the context of convex, starlike and prestarlike functions in the unit disc on many occasions, see for instance Lewis [11] and Ruscheweyh [15]. Since these functions are in $\mathcal{H}(\Lambda)$ we again can ask which of them, after re-normalization, are also universally starlike or convex. Our answers here are not complete (which cannot be expected, anyway).

Theorem 1.8. *Let* $c > 0$ *and* $0 \leq a \leq \min\{1,c\},\ 0 \leq b \leq c$. *Then the function* $f(z) := z\,{}_2F_1(a,b,c,z)$ *is universally starlike.*

Using the first invariance property stated in Theorem 1.4 together with Euler's transformation formula for hypergeometric functions (cf. [1, 15.3.4]) Theorem 1.8 implies

$$z(1-z)^{a-1}{}_2F_1(a,b,c,z) \in \mathcal{R}^{\mathrm{u}}_{1/2}, \quad a,b,c > 0,\ a \leq \min\{1,c\}.$$

Theorem 1.9. *Let* $a,b,c \neq 0$ *and* $-1 \leq a \leq \min\{1,c\},\ -1 \leq b \leq c$. *Then the function* $f(z) = \frac{c}{ab}({}_2F_1(a,b,c,z) - 1)$ *is universally convex.*

From the large number of special cases we just mention the following:

$$\begin{aligned} \sqrt{z}\arcsin\sqrt{z} &= z{}_2F_1(\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{2},z), \\ \frac{z}{(1-z)^a} &= z{}_2F_1(a,1,1,z), \quad 0 < a \leq 1, \end{aligned}$$

are universally starlike, while, for instance,

$$u_a(z) := \frac{1}{a}(1-(1-z)^a) = \frac{1}{a}(1 - {}_2F_1(-a,1,1,z)), \quad 0 \neq a^2 \leq 1,$$

$$\log\frac{1}{1-z} = \lim_{a\to 0} u_a(z), \tag{1.6}$$

are universally convex. It might be interesting to note that the inverse $w = F(z)$ of the Koebe function $z = 4w(1+w)^{-2}$, namely

$$\frac{2}{z}(1-\sqrt{1-z}) - 1 = {}_2F_1(\tfrac{1}{2},1,2,z) - 1 = \frac{z}{4} + \dots,$$

is also universally convex (up to the scale factor $\frac{1}{4}$).

One should remark that there are a number of theorems by various authors (for instance, Van Vleck [21], Hurwitz [8], Schafheitlin [17], Runckel [14] and Küstner [9]) dealing with the non-vanishing of hypergeometric functions in Λ. Our results above concern univalence in Λ, which, at least for the derivatives, imply non-vanishing statements as well, and the corresponding parameter sets $\{a, b, c\}$ have large intersections. We refer to Küstner [9] for more about this.

1.7. **A conjecture.** There are many explicit examples of completely monotone sequences, and it is an interesting question whether their corresponding power series (multiplied by z) belong to $\mathcal{R}_\alpha^u$ for some $\alpha < 1$. A somewhat curious example comes from a famous problem due to Ramanujan. For $n = 0, 1, 2, \dots$ define the numbers θ_n by

$$\frac{e^n}{2} = \sum_{k=0}^{n-1} \frac{n^k}{k!} \; + \; \frac{n^n}{n!}\theta_n.$$

Ramanujan's question was whether $\theta_n \in [1/3, 1/2]$ holds for all n. This has been proved on several occasions (see Berndt [4] for details). Only recently, Adell & Jodrá [2] showed that the sequence θ_n is in fact completely monotone, with $\theta_n \to \frac{1}{3}$ for $n \to \infty$. Numerical as well as graphical experiments, using a new explicit representation for the θ_n given in [2] seem to indicate that the function

$$\sigma(z) := \frac{1}{\theta_1 - \frac{1}{3}} \sum_{n=1}^{\infty} (\theta_n - \frac{1}{3}) z^n$$

is indeed universally convex. This is an open problem. One can show, however, that σ cannot be universally prestarlike of any order smaller than $-.05$.

1.8. **A useful theorem.** The definition of universally prestarlike functions deals generally with the question when the quotient of two functions belongs to $\mathcal{T}$. In this context the following Theorem proves to be useful. It is also interesting by itself.

Theorem 1.10. *Let $f, g \in \mathcal{T}$ be represented by*

$$f(z) = \int_0^1 \frac{\varphi(t)\,dt}{1 - tz}, \quad g(z) = \int_0^1 \frac{\psi(t)\,dt}{1 - tz}$$

for non-negative Borel functions φ and ψ on $(0, 1)$. If $\varphi(t)\psi(s) \geq \varphi(s)\psi(t)$ holds for $0 < s \leq t < 1$, then $f(z)/g(z) \in \mathcal{T}$.

2. Proof of Theorem 1.1 and its Corollaries

2.1. **A characterization of members of $\mathcal{T}$.** Later we will need the following independent characterization of the functions belonging to the family $\mathcal{T}$, introduced in Section 1.1.

Lemma 2.1. *Let $F \in \mathcal{H}(\Lambda)$. Then $F \in \mathcal{T}$, i.e.*

$$F(z) = \int_0^1 \frac{d\mu(t)}{1 - tz}$$

for some probability measure μ on $[0, 1]$, if and only if the following conditions are fulfilled:

(i) $F(0) = 1$;

(ii) $F(x) \in \mathbb{R}$ *for* $x \in (-\infty, 1)$;

(iii) $\operatorname{Im} F(z) \geq 0$ *for* $\operatorname{Im} z > 0$;

(iv) $\lim_{n\to\infty} F(z_n)/z_n = 0$ *for some sequence* $z_n \in \mathbb{C}$ *with* $\operatorname{Im} z_n \to +\infty$, *and* $\operatorname{Im} z_n \geq \delta \operatorname{Re} z_n$ *for some positive constant* δ*;*

(v) $\limsup_{x\to\infty} F(-x) \geq 0$.

The measures μ *and the functions* F *are in one-to-one correspondence.*

This lemma is not really new: bits and pieces of it are available in the literature, see for instance the monographs [3], [5], [20]. Since we need it in exactly this form we prefer to sketch the less common parts of it.

Proof. The 'only if' part of the Lemma is more or less obvious: note that (iv) and (v) follow (almost) immediately from

$$\lim_{r\to+\infty} F(r\zeta) = \lim_{r\to+\infty} \int_0^1 \frac{d\mu(t)}{1-tr\zeta} = \mu(\{0\}) \in [0,1],\ \zeta \in \mathbb{C}\setminus[0,\infty], \tag{2.1}$$

using Lebesgue's dominated convergence theorem.

To prove the other direction we first note that (ii), (iii) imply that F is a Pick function in $\mathcal{H}(\Lambda)$ which is real on $(-\infty,1)$, i.e. a member of the class $P(-\infty,1)$ in the notation of [5, II, Lemma 2]. Hence, it has a representation ([5, II, Theorem 1], [20, p.23])

$$F(z) = a + bz + \int_1^\infty \frac{1+uz}{u-z} d\sigma(u), \quad z \in \Lambda,$$

where $a \in \mathbb{R}$, $b \geq 0$, and σ is a finite non-negative Borel measure on $[1,\infty)$. Using Lebesgue's dominated convergence theorem, in combination with (iv), for a sequence with $\operatorname{Im} z_n \geq \delta \operatorname{Re} z_n$ and $\operatorname{Im} z_n \to +\infty$ we immediately deduce: $b = 0$, and condition (i) yields

$$a = 1 - \int_1^\infty \frac{d\sigma(u)}{u}.$$

Therefore, under the assumptions (i) through (iv), we obtain the representation

$$F(z) = 1 + \int_1^\infty \left(\frac{1+uz}{u-z} - \frac{1}{u}\right) d\sigma(u) = 1 + z\int_1^\infty \frac{1+u^2}{u(u-z)} d\sigma(u).$$

From this representation we immediately deduce that $F(x)$ is non-decreasing in $x < 1$, so that $c := \lim_{x\to-\infty} F(x)$ exists (and is in [0,1] by (v)). By Lebesgue's monotone convergence theorem, we have

$$c = 1 - \lim_{n\to+\infty} \int_1^\infty \frac{x_n(1+u^2)}{u(u+x_n)} d\sigma(u) = 1 - \int_1^\infty \frac{1+u^2}{u} d\sigma(u).$$

Thus, the positive measure λ on $[1,\infty]$ defined by $d\lambda(u) = (u + 1/u)d\sigma(u)$ has the total mass $1 - c \in [0,1]$ and we have

$$F(z) = 1 + z\int_1^\infty \frac{d\lambda(u)}{u-z}.$$

Using the substitution $t = \frac{1}{u}$ we obtain

$$F(z) = 1 + z\int_0^1 \frac{t\,d\mu^*(t)}{1-tz},$$

where $d\mu^*(t) = d\lambda(u)$ on $t \in [0,1]$. Clearly, the total masses of $\mu*$ and λ are equal, namely $1-c$. We now define $d\mu(t) := d\mu^*(t) + c\, d\delta_0(t)$, where δ_0 stands for the Dirac measure at 0. Then μ is a probality measure on $[0,1]$, and we have

$$F(z) = 1 + z\int_0^1 \frac{t\, d\mu(t)}{1-tz} = \int_0^1 \frac{d\mu(t)}{1-tz},$$

the assertion. The uniqueness of the measure μ for the given F follows from the fact that the Hausdorff moment sequences are determinate, see [20, p.23] . $\square$

2.2. **Two basic lemmas.** The essential link between universally prestarlike functions and Pick functions, as described in the Lemma 2.1, is contained in the following elementary lemmas.

Lemma 2.2. *Let $F \in \mathcal{H}(\Lambda)$ be such that*

$$\operatorname{Re}\left\{\frac{1}{1-z}F\left(\frac{\gamma z}{1-z}\right)\right\} \geq 0, \quad |z|<1,\ |\gamma+1|<1. \tag{2.2}$$

Then F is a Pick function of class $P(-\infty,1)$, i.e. F fulfills the conditions (ii),(iii) of Lemma 2.1.

Proof. Let w satisfy $\operatorname{Im} w > 0$. Then, for $\gamma := \frac{w}{i\alpha}$ with $\alpha < 0$ and $|\alpha|$ large enough we have $|\gamma+1|<1$. Clearly, condition (2.2) holds also for $|z|=1, z\neq 1$, so that we can choose z on that circle such that

$$\frac{1}{1-z} = \frac{1}{2} + i\alpha, \qquad \frac{z}{1-z} = -\frac{1}{2} + i\alpha.$$

Hence, after division by α, we get

$$\operatorname{Re}\left\{\left(\frac{1}{2\alpha}+i\right)F\left(\frac{w}{i\alpha}\left(-\frac{1}{2}+i\alpha\right)\right)\right\} \leq 0,$$

or, by letting $\alpha \to -\infty$, that $\operatorname{Im} F(w) \geq 0$. Similarly we find $\operatorname{Im} F(w) \leq 0$ for $\operatorname{Im} w < 0$, and, for continuity reasons, $F(x) \in \mathbb{R}$ for $x \in (-\infty,1)$. $\square$

Lemma 2.3. *Let $\beta \geq 0$ and $f \in \mathcal{H}(\Lambda)$. Then, for each choice of $\rho \in [0,1]$ and $\gamma \in \mathbb{C}$ with $|\gamma+\rho| \leq 1$, the following relation holds (recall the notation from Definition 1.1)*

$$(D^\beta f_{\gamma,\rho})(z) = (1-\rho z)^{1-\beta}(D^\beta f)_{\gamma,\rho}(z). \tag{2.3}$$

In particular we have

$$\frac{(D^{\beta+1} f_{\gamma,\rho})(z)}{(D^\beta f_{\gamma,\rho})(z)} = \frac{1}{1-\rho z}\frac{(D^{\beta+1} f)_{\gamma,\rho}(z)}{(D^\beta f)_{\gamma,\rho}(z)} \tag{2.4}$$

.

Proof. Let $w(z) := \frac{\gamma z}{1-\rho z}$ and $J_\beta(z) := \frac{z}{(1-z)^\beta}$. We apply the following transformations:

$$\begin{aligned}
\gamma\,(D^\beta f_{\gamma,\rho})(z) &= J_\beta(z) *_z f(w(z)) \\
&= J_\beta(z) *_z \left\{ \frac{1}{1-\zeta w(z)} *_\zeta f(\zeta) \right\}\Big|_{\zeta=1} \\
&= J_\beta(z) *_z \left(\left(\frac{\gamma\zeta}{\gamma\zeta+\rho} \frac{1}{1-(\gamma\zeta+\rho)z} + \frac{\rho}{\gamma\zeta+\rho} \right) *_\zeta f(\zeta) \right)\Big|_{\zeta=1} \\
&= \frac{\gamma\zeta z}{(1-z(\gamma\zeta+\rho))^\beta} *_\zeta f(\zeta)\Big|_{\zeta=1} \\
&= (1-\rho z)^{1-\beta} \left(J_\beta(w(z)\zeta) *_\zeta f(\zeta) \right)\big|_{\zeta=1} \\
&= \gamma\,(1-\rho z)^{1-\beta} (D^\beta f)_{\gamma,\rho}(z),
\end{aligned}$$

where a symbol $*_\sigma$ indicates convolution w.r.t. the variable σ. Taking quotients of the cases $\beta+1$ and β of (2.3) gives (2.4). □

2.3. **Proof of Theorem 1.1.** Let $\beta = 2 - 2\alpha$ so that $\beta \geq 0$, and let $f \in \mathcal{H}_1(\Lambda)$. First assume that $f \in \mathcal{R}^{\mathrm{u}}_\alpha$. By definition this is equivalent to the statement that the left hand side of (2.4) is in $\mathcal{P}$ for all $\rho \in [0,1]$, $|\gamma+\rho| \leq 1$.

Now let

$$F(w) := \frac{(D^{\beta+1} f)(w)}{(D^\beta f)(w)}. \tag{2.5}$$

Clearly $F \in \mathcal{H}(\Lambda)$, and by Lemma 2.3 we find

$$\operatorname{Re} \frac{1}{1-\rho z} F\left(\frac{\gamma z}{1-\rho z} \right) \geq \frac{1}{2}, \quad z \in \mathbb{D}, \tag{2.6}$$

with ρ, γ as before.

Lemma 2.2 then shows that F satisfies the conditions (i)-(iii) of Lemma 2.1. To prove the "only if" part of Theorem 1.1 we are left with the verification of conditions (iv) and (v).

From (2.6) with $\gamma = -1$ and $\rho = 1$ we have the subordination

$$\frac{1}{1-z} F\left(\frac{-z}{1-z} \right) \prec \frac{1}{1-z},$$

which implies

$$\frac{1}{1+z} \leq \frac{1}{1-z} F\left(\frac{-z}{1-z} \right), \quad z \in (0,1),$$

and therefore (v) by letting $z \to 1-0$.

To prove (iv) we set again $\rho = 1$, $z = r \in (0,1)$, and $\gamma = \gamma(r) := r - 1 + i\sqrt{1-r^2}$, so that $|\gamma(r)+1| = 1$. By the same subordination argument as above we obtain with

$$w(r) = \frac{\gamma(r) r}{1-r} = -1 + i\sqrt{\frac{1+r}{1-r}}$$

the inequality

$$\left| \frac{1}{1-r} F(w(r)) \right| \leq \frac{1}{1-r}.$$

Since $\operatorname{Re} w(r) = -1, \operatorname{Im} w(r) \to \infty$ and $F(w(r))/w(r) \to 0$ for $r \to 1-0$ condition (iv) is also established.

If, on the other hand, the function $F(w)$, as given in (2.5), belongs to $\mathcal{T}$ then, by Lemma 2.3, we have

$$\frac{(D^{\beta+1}f_{\gamma,\rho})(z)}{(D^{\beta}f_{\gamma,\rho})(z)} = \int_0^1 \frac{d\mu(t)}{(1-\rho z)(1-w(z)t)} = \int_0^1 \frac{d\mu(t)}{1-(\rho+\gamma t)z}$$

for the admissible parameters ρ, γ and the probability measure μ corresponding to $F \in \mathcal{T}$. But since $|\rho+\gamma t| \le |(\rho+\gamma)t| + (1-t)|\rho| \le 1$ for $t \in [0,1]$ it is clear that the function on the right is indeed in $\mathcal{P}$, which we had to establish. □

2.4. **Proof of Corollary 1.1.** We first remark that

$$D^{3-2\alpha}f = \frac{1-2\alpha}{2-2\alpha}(D^{2-2\alpha}f) + \frac{1}{2-2\alpha}z(D^{2-2\alpha}f)',$$

so that (1.4) implies

$$\frac{z(D^{2-2\alpha}f)'}{D^{2-2\alpha}f} - \frac{1}{z} = (2-2\alpha)\int_0^1 \frac{t}{1-tz}d\mu(t),$$

and therefore, after some calculation, the assertion. □

2.5. **Proof of Corollary 1.2.** The inclusion property in Lemma 1.1 together with Definition 1.2 imply $\mathcal{R}^{\mathrm{u}}_{\alpha} \subset \mathcal{R}^{\mathrm{u}}_{\beta}$ for $\alpha \le \beta \le 1$. The assertion follows now from Theorem 1.1. □

2.6. **Proof of Corollary 1.3.** Using a discrete measure in Corollary 1.1 the assertion follows from Corollary 1.2, applied to the corresponding $f \in \mathcal{R}^{u}_{1-\sigma/2}$. □

3. Proof of Theorems 1.2 – 1.4

3.1. **Proof of Theorem 1.2.** We only need to deal with the case $\alpha \le \frac{1}{2}$. By construction and a limiting argument, f maps the upper half-plane univalently onto a domain starlike w.r.t. the origin, and is also located in the upper half-plane (note that f is typically real). The image of the lower half-plane is obtained by reflection at the real axis. Hence f is univalent in Λ, and the image is starlike w.r.t. the origin. □

3.2. **Proof of Theorem 1.3.** Assume $f \in \mathcal{R}^{\mathrm{u}}_{1/2}$ is rational. Since $f/z \in \mathcal{T}$, we must have

$$f(z) = \sum_{k=1}^{n} \frac{\mu_k z}{1-t_k z}, \qquad t_k \in [0,1],\ \mu_k > 0,\ \sum_{k=1}^{n} \mu_k = 1.$$

For a fixed k, which we may choose as $k = 1$, we then have

$$f(z) = \frac{\mu_1 z}{1-t_1 z} + H(z),$$

where H is rational and analytic in $z = 1/t_1$. Then a simple calculation yields

$$F(z) := \frac{zf'(z)}{f(z)} = \frac{1}{1-t_1 z} + H^*(z), \tag{3.1}$$

where H^* is also analytic in $z = 1/t_1$. But, by Theorem 1.1, F is also in $\mathcal{T}$ and is clearly also a rational function. So it must also be of the form

$$\sum_{k=1}^{m} \frac{\lambda_k}{1-\tau_k z}, \qquad \tau_k \in [0,1],\ \lambda_k > 0,\ \sum_{k=1}^{m} \lambda_k = 1.$$

If such a function has the form (3.1), then one of the τ_k must equal t_1, and the corresponding λ_k must be 1. But then we must have $F(z) = zf'(z)/f(z) = \frac{1}{1-t_1 z}$. Integration shows that $f(z) = \frac{z}{1-t_1 z}$. □

3.3. **Proof of Theorem 1.4.** Let $F(z) := -f(z/(z-1))$. We have to show that

$$F_{\gamma,\rho}(z) = \frac{-1}{\gamma} f\left(\frac{-\gamma z}{1-(\gamma+\rho)z}\right) = f_{-\gamma,\gamma+\rho}(z) \in \mathcal{R}_\alpha, \quad |\gamma+\rho| \le 1.$$

This is true for $\gamma+\rho = 0$. Otherwise we set $x := (\overline{\gamma+\rho})/|\gamma+\rho|$ and find

$$\frac{1}{x} f_{-\gamma,\gamma+\rho}(xz) = f_{-\gamma x,|\gamma+\rho|}(z),$$

which belongs to $\mathcal{R}_\alpha$ since $||\gamma+\rho| - \gamma x| = ||\gamma+\rho|\bar{x} - \gamma| = |\rho| \le 1$ and $|\gamma+\rho| \ge 0$. The assertion about $f(tz)/t$ can be established in a similar fashion. □

4. Proof of Theorems 1.5 and 1.6

4.1. **Proof of Theorem 1.5.** Let f be universally starlike. Following the proof of Theorem 1.1 with the required changes (choose $\beta = 1$) it follows immediately that

$$\mathrm{Re}\left\{\frac{1}{1-\rho z}\frac{w(z)f'(w(z))}{f(w(z))}\right\} > 0$$

for $z \in \mathbb{D}$, where $w(z), \gamma, \rho$ are as before. This, together with Lemma 2.2, implies that

$$F(w) := \frac{wf'(w)}{f(w)}$$

satisfies the conditions (i)-(iii) of Lemma 2.1, and the same argument as in the proof of Theorem 1.1 yields the validity of (iv) for F as well. To prove (v) let

$$c := \lim_{x\to\infty, x\in\mathbb{R}} F(-x).$$

Since

$$\frac{1}{1-z} F\left(\frac{-z}{1-z}\right) \prec \frac{1+z}{1-z}$$

and consequently

$$\frac{1-z}{1+z} \le \frac{1}{1-z} F\left(\frac{-z}{1-z}\right), \quad z \in (0,1),$$

so that $c \ge 0$ by letting $z \to 1-0$. Thus $F \in \mathcal{T}$ by Lemma 2.1, and Theorem 1.1 shows that f is universally prestarlike of order $\frac{1}{2}$. □

4.2. **A general property of convex univalent maps.** The following lemma is well-known. Discs and half-planes are called circular domains.

Lemma 4.1. *Let Ω' be a circular subdomain of a circular domain $\Omega \subset \mathbb{C}$. If $f \in \mathcal{H}(\Omega)$ maps Ω conformally onto a convex domain then $f(\Omega')$ is also convex.*

The proof is elementary. See Pommerenke [13], Heins [7] and Sheil-Small [19] for even more general statements which could be used to extend Theorem 1.6.

4.3. **Proof of Theorem 1.6.** If f is universally convex, Lemma 4.1 shows that f maps every circular domain in Λ onto a convex domain. This holds, in particular, for the domains $\Omega_{\gamma,\rho}$ with $|\gamma+\rho| \le 1$ and the corresponding functions $f_{\gamma,\rho}$ are therefore members of $\mathcal{R}_0$. This implies $f \in \mathcal{R}_0^{\mathrm{u}}$. The other direction is immediate. □

5. Proof of Theorem 1.7 and 1.10

5.1. Our proof of Theorem 1.7 uses the following lemma, which in turn is based on Theorem 1.10.

Lemma 5.1. *Let g_α be as in* (1.5)*. Then we have*

$$\frac{g_\alpha(z)}{g_\beta(z)} \in \mathcal{T}, \quad 0 \le \alpha \le \beta. \tag{5.1}$$

5.2. **Proof of Theorem 1.10.** Let $h(z) = f(z)/g(z), z \in \Lambda$. We check conditions (i)-(v) in Lemma 2.1 for h. Conditions (i) and (ii) clearly hold. For (iv), we look at the quantity

$$\frac{h(iy)}{y} = \frac{f(iy)}{yg(iy)}$$

for $0 < y \to +\infty$. Obviously, the numerator tends to zero, and

$$\lim_{y\to+\infty} iy\, g(iy) = \int_0^1 \frac{iy\psi(t)dt}{1-ity} \to -\int_0^1 \frac{\psi(t)dt}{t},$$

does not vanish (and might be ∞). Therefore, condition (iv) has been confirmed.

We next proceed to (iii). Note first that $\overline{h(z)} = h(\overline{z})$ by (ii). Since

$$2i \operatorname{Im} h(z) = h(z) - h(\overline{z}) = \frac{f(z)g(\overline{z}) - f(\overline{z})g(z)}{|g(z)|^2},$$

we have only to look at the numerator of the last term. Since

$$\begin{aligned}
f(z)g(\overline{z}) &= \int_0^1 \frac{\varphi(s)ds}{1-sz} \int_0^1 \frac{\psi(t)dt}{1-t\overline{z}} \\
&= \iint \frac{\varphi(s)\psi(t)}{(1-sz)(1-t\overline{z})} dsdt \\
&= \iint_{s\le t} + \iint_{t \le s} \\
&= \iint_{s\le t} \left(\frac{\varphi(s)\psi(t)}{(1-sz)(1-t\overline{z})} + \frac{\varphi(t)\psi(s)}{(1-tz)(1-s\overline{z})} \right) dsdt,
\end{aligned}$$

we obtain

$$f(z)g(\bar{z}) - f(\bar{z})g(z) = \iint_{\{0<s\le t<1\}} \frac{(t-s)(\varphi(t)\psi(s) - \varphi(s)\psi(t))(z-\bar{z})}{|1-sz|^2|1-tz|^2} dsdt.$$

By assumptions, (iii) follows. (v) is immediate since $h(x) = f(x)/g(x) > 0$ for $x \in (-\infty, 1)$, $\square$

5.3. **Proof of Lemma 5.1.** Lemma 5.1 follows immediately from Theorem 1.10, given the expressions for g_α and g_β in (1.5). $\square$

5.4. **Proof of Theorem 1.7.** The starlike case is equivalent to

$$\frac{z\mathrm{Li}'_\alpha(z)}{\mathrm{Li}_\alpha(z)} = \frac{\mathrm{Li}_{\alpha-1}(z)}{\mathrm{Li}_\alpha(z)} = \frac{g_{\alpha-1}(z)}{g_\alpha(z)} \in \mathcal{T}$$

which, for $\alpha \geq 1$, is an immediate consequence of Lemma 5.1. The case $\alpha = 0$ is obvious.

In the convex case we can write

$$1 + \frac{z}{2}\frac{\mathrm{Li}''_\alpha(z)}{\mathrm{Li}'_\alpha(z)} = \frac{1}{2}\left(1 + \frac{\mathrm{Li}_{\alpha-2}(z)}{\mathrm{Li}_{\alpha-1}(z)}\right) = \frac{1}{2}\left(1 + \frac{g_{\alpha-2}(z)}{g_{\alpha-1}(z)}\right).$$

By Lemma 5.1 we have $g_{\alpha-2}/g_{\alpha-1} \in \mathcal{T}$ for $\alpha \geq 2$, and the assertion follows from the fact that $1 \in \mathcal{T}$ and $\mathcal{T}$ is a convex family. The case $\alpha = 1$ comes from (1.6). $\square$

6. Proof of Theorems 1.8 and 1.9

We begin with the following result by Küstner [9, Thm. 1.5].

Lemma 6.1. *Let $-1 < a < c$ and $0 < b < c$. Then*

$$\frac{{}_2F_1(a+1,b,c,z)}{{}_2F_1(a,b,c,z)} \in \mathcal{T}.$$

6.1. **Proof of Theorem 1.8.** For the function f we find

$$\begin{aligned} z\frac{f'(z)}{f(z)} &= 1 + \frac{z\ {}_2F_1'(a,b,c,z)}{{}_2F_1(a,b,c,z)} \\ &= 1 + \frac{ab}{c}\ \frac{z\ {}_2F_1(a+1,b+1,c+1,z)}{{}_2F_1(a,b,c,z)} \\ &= 1 - a + a\ \frac{{}_2F_1(a+1,b,c,z)}{{}_2F_1(a,b,c,z)} \in \mathcal{T}. \end{aligned}$$

Here we made use of the identity

$$\frac{b}{c}\ z\ {}_2F_1(a+1,b+1,c+1,z) = {}_2F_1(a+1,b,c,z) - {}_2F_1(a,b,c,z),$$

which is easily verified, and of Lemma 6.1, combined with the convexity of $\mathcal{T}$. $\square$

6.2. **Proof of Theorem 1.9.** With the same reasoning we obtain for the function f of this theorem that

$$\begin{aligned} 1 + \frac{z}{2}\frac{f''(z)}{f'(z)} &= 1 + \frac{z}{2}\ \frac{(a+1)(b+1)}{(c+1)}\ \frac{{}_2F_1(a+2,b+2,c+2,z)}{{}_2F_1(a+1,b+1,c+1,z)} \\ &= 1 - \frac{a+1}{2} + \frac{a+1}{2}\ \frac{{}_2F_1(a+2,b+1,c+1,z)}{{}_2F_1(a+1,b+1,c+1,z)} \in \mathcal{T}. \end{aligned}$$

$\square$

References

[1] M. Abramowitz and I.A. Stegun, editors, Handbook of mathematical functions, Dover, New York, 1970.

[2] J.A. Adell and P. Jodrá, On the complete monotonicity of a Ramanujan sequence connected with e^n, Pre-Publicaciones Sem. Mat. Zaragoza **5** (2005).

[3] N. I. Akhiezer and I. M. Grazman, Theory of Linear Operators in Hilbert Space, 2 vols., Frederic Ungar Publ., New York, 1963, Translated from the Russian by M. Nestell.

[4] B.C. Berndt, Ramanjan's notebooks, Part II, Springer-Verlag, 1989.

[5] W.B. Donoghue Jr., Monotone matrix functions and analytic continuation, Grundlehren math. Wiss. **207**, Springer Verlag (1974)

[6] F. Hausdorff, Summationsmethoden und Momentfolgen I., Math. Zeitschr. **9** (1921), 74–109.

[7] M. Heins, On a theorem of Study concerning conformal maps with convex images, Mathematical Essays dedicated to A. J. MacIntyre, Hari Shankar *ed.* , Ohio University Press, 1970

[8] A. Hurwitz, Über die Nullstellen der hypergeometrischen Funktion, Math. Ann. **64** (1907), 517–560.

[9] R. Küstner, Mapping properties of hypergeometric functions and convolutions of starlike or convex functions of order α, Comp. Meth. Funct. Theory **2** (2002), 597–610.

[10] J. Lewis, Convexity of a certain series, J. Lond. Math. Soc., II. Ser. **27** (1983), 435–446.

[11] J. Lewis, Applications of a convolution theorem to Jacobi polynomials, SIAM J. Math. Analysis, **10** (1979), 1110–1120.

[12] G. Pólya and I.J. Schoenberg, Remarks on de la Vallée Poussin means and convex conformal maps of the circle, Pacific J. Math., **8** (1958), 295-334.

[13] Chr. Pommerenke, Images of convex domains under convex conformal mappings, Michigan Math. J., **9** (1962), 257

[14] H.-J. Runckel, On the zeros of the hypergeometric function, Math. Ann. **191** (1971), 53–58.

[15] S. Ruscheweyh, Convolutions in geometric function theory, Sem. Math. Sup. **83**, Les Presses de l'Université de Montréal (1982).

[16] S. Ruscheweyh, T.B. Sheil-Small, Hadamard products of schlicht functions and the Pólya-Schoenberg conjecture, Commentarii Math. Helvetici **48** (1973), 119–135.

[17] P. Schafheitlin, Die Nullstellen der hypergeometrischen Funktion, S.-Ber. Berl. Math. Ges. **7** (1908), 19–28.

[18] T.B. Sheil-Small, Complex Polynomials, Cambridge Stud. Adv. Math. **75**, (2002).

[19] T.B. Sheil-Small, The Hadamard product and linear transformations of classes of analytic functions, J. Analyse Math. **34** (1978), 204–239.

[20] J.A. Shohat, J.D. Tamarkin, The Problem of Moments, Amer. Math. Soc. (1943).

[21] E.B. van Vleck, A determination of the number of real and imaginary roots of the hypergeometric series, Trans. Amer. Math. Soc. **3** (1902), 110–131.

[22] K.J. Wirths, Über totalmonotone Zahlenfolgen, Arch. Math. **26** (1975), 508–517.

Mathematisches Institut, Universität Würzburg, D-97074 Würzburg, Germany
E-mail address: `ruscheweyh@mathematik.uni-wuerzburg.de`

Departamento de Informática, Universidad Técnica F. Santa María, Valparaíso, Chile
E-mail address: `lsalinas@inf.utfsm.cl`

Department of Mathematics, Graduate School of Science, Hiroshima University, Higashi-Hiroshima, 739-8526 Japan
E-mail address: `sugawa@math.sci.hiroshima-u.ac.jp`