POINCARÉ METRIC AND THE INDUCED DISTANCE OF PLANE DOMAINS

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The present talk is partially based on the joint work [3] with Matti Vuorinen.

The hyperbolic (or *Poincaré*) metric $\rho_\Omega(z)|dz|$ of a (hyperbolic) plane domain $\Omega \subset \mathbb{C}$ is defined by $\rho_\Omega(p(\zeta))|p'(\zeta)| = 1/(1-|\zeta|^2)$, where p is a holomorphic universal covering map of Ω from the unit disk $\mathbb{D} = \{\zeta \in \mathbb{C}; |\zeta| < 1\}$. The hyperbolic distance $d_\Omega(z_1, z_2)$ of a pair of points z_1 and z_2 in Ω is defined to be the infimum of hyperbolic lengths of rectifiable curves joining them in Ω.

In general, it is difficult or almost impossible to obtain a concrete form of the holomorphic universal covering map of a given domain. Consequently, only few domains admit us to compute the hyperbolic density $\rho_\Omega(z)$ or distance $d_\Omega(z_1, z_2)$. Known examples are essentially simply or doubly connected domains, twice punctured planes, and their corvering domains.

On the other hand, it is important to know values of the hyperbolic density of a domain in various situations such as Schottky's theorem and Littlewood's theorem. Therefore, the last resort is to estimate the hyperbolic density or distance.

In this talk, we collect standard cases where the hyperbolic density can be given in a concrete way. Then, by using them, we introduce several approaches to the estimation of hyperbolic metric. In particular, we give a simple proof of a recent result of Gardiner and Lakic in [1]. We also propose a probably new quantity

$$m(a,t) = \inf_{b \in \partial\Omega} \big| t - \log|a-b| \big|,$$

which is Lipschitz continuous with bound 1 in t, to measure the magnitude of clustering of $\partial\Omega$ around a boundary point $a \in \partial\Omega$.

Finally, we give an effective way to estimate the hyperbolic distance from below. The idea goes back to Hayman [2].

References

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