UNIFORMLY PERFECT SETS
— ANALYTIC AND GEOMETRIC ASPECTS —

TOSHIYUKI SUGAWA

1. Definition and background of uniform perfectness

The classical analysis had mainly treated smooth objects. Recently, however, it has been recognized that non-smooth ones such as self-similar sets could have relatively simple structures, and therefore, they are also natural objects in Mathematics. For those sets which have a sort of self-similarity, one has been able to develop deep analysis by using their combinatorial structures in recent years. This methodology is quite powerful, however, it is hard to develop a unified approach because the method strongly depends on the combinatorial structure. Therefore, if one could invent a mathematical quantity which is applicable to general compact sets, is easy to treat, reflects well a specific property of the figures and is informative, then it would play an important role in the progress of this field.

For a connected compact set containing at least two points, which is usually called a continuum, the linear projection of the set into a suitable lower dimensional space can be used to obtain useful information. On the other hand, an isolated point is quite simple to handle, and hence, can be excluded from our considerations at the moment. A non-empty closed set without any isolated point is called perfect in the theory of general topology. Hence, one may think that a perfect set in the Euclidean space containing no continuum, which is called a (generalized) Cantor set, is most difficult to treat in some sense.

In this article, we give an exposition for the notion of uniform perfectness, which is a quantified version of perfectness, and, in spite of the apparent simplicity of its definition, we will see that the uniform perfectness yields important information on the metric structure of the set even in the case when it is totally disconnected. As we will see in Section 5, this property is indeed enjoyed by most of compact sets which have certain self-similarity. In the sequel, in order to collect various interesting aspects as many as possible, we restrict ourselves to the case of compact sets in the complex plane or in the Riemann sphere (2-sphere). For a general case, we only refer to the present state of the study in § 6.1. To avoid the trivial case, in what follows, we consider only compact sets containing at least two points unless otherwise stated.

1.1. **Definition.** A compact set E in the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is said to be *uniformly perfect* if there exists a constant $c > 0$ with the following property: For any finite point $a \in E$ and for any $0 < r < d(E)$, there is a point $b \in E$ with $cr \le |b - a| \le r$. Here, $d(E)$ denotes the Euclidean diameter of the set E and we conventionally set $d(E) = +\infty$ whenever $\infty \in E$. Then, since we can take a sequence in $E \setminus \{a\}$ tending to a, we see that E is perfect. The above definition says that if we look at E around a through a

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microscope with any magnification we can still find another point in E within a definite interval of the radius. In this sense, the word "uniform" has been used. As one can easily see from the definition, this concept is invariant under similarities of the plane. In fact, (if we allow the change of the above constant c) we will find that it is invariant under more general transformations on $\widehat{\mathbb{C}}$ including Möbius maps (see § 2.1). With this in mind, the reader may feel it strange that the point at infinity plays a special role in the above definition of the uniform perfectness. Through more natural formulations of the uniform perfectness below, however, the universality of this concept will be clarified.

1.2. **Historical remarks.** The uniform perfectness is quite simple concerning the definition, however, as far as the author knows, this terminology first appeared in Pommerenke [61], 1979, although the same condition was stated in Beardon-Pommerenke [8], 1978, earlier. Note that, at the almost same time, the essentially same concept was given by Tukia-Väisälä [81] under the name "homogeneously dense". It might be a surprising fact that such a simple notion had not been systematically studied before then. As a fact, once the usefulness of this notion has been recognized, many authors began to investigate the properties more deeply and to apply it. In these two decades, various connections between the uniform perfectness and other fields have emerged. The recent increase of interest in the fractal geometry has produced the demand for tools to measure mathematical objects which are not necessarily smooth. The uniform perfectness could provide an effective means as such a tool.

1.3. **Personal background.** The author would like to note here his personal motivation to choose the uniform perfectness as one of the research subjects in these years. Originally, in order to investigate the Bers embedding of Teichmüller spaces, he had been struggling to understand what the Schwarzian derivative was in his way. For a certain aim, he considered a condition for the hyperbolic sup-norm of the Schwarzian derivative of analytic universal covering map of a given domain to be finite, and then, arrived at the notion of uniform perfectness around 1994. He had obtained a several equivalent properties to the uniform perfectness, however, he found later that Pommerenke had done almost all of them and even much more in the series of his work. Since the author had also obtained new or more refined results in this direction, he decided to publish it as a survey [72].

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2. Geometric characterizations
– rear view of uniform perfectness –

Denote by Ω the complement of a compact set E in $\widehat{\mathbb{C}}$. We will give characterizations of the uniform perfectness of E in terms of the geometry of the open set Ω. Therefore, we will mainly be concerned with the complement in this section. Note that the boundary $\partial\Omega$ is taken in the Riemann sphere. So, the relative boundary $\partial\Omega \setminus \{\infty\}$ of Ω in $\mathbb{C}$ will be distinguished by the different notation $\partial_{\mathrm{b}}\Omega$ here.

2.1. **Characterizations in terms of moduli of annuli.** In this article, a doubly connected domain will be called an *annulus*, and an annulus of the form $r_1 < |z-a| < r_2$ will be called a *round annulus*. An arbitrary annulus A is known to be conformally equivalent to a round annulus. Then the quantity $\log(r_2/r_1)$ is a conformal invariant determined by A only, called the *modulus* of A, and denoted by $\operatorname{mod} A$. An annulus A in Ω is said to *separate* the boundary $\partial\Omega$ if each connected component of $\widehat{\mathbb{C}} \setminus A$ intersects $\partial\Omega$. This is equivalent to saying that A separates $E = \widehat{\mathbb{C}} \setminus \Omega$. Then we have the following.

Theorem 2.1 ([62], [72]). *The uniform perfectness of $E = \widehat{\mathbb{C}} \setminus \Omega$ is equivalent to each of the following conditions:*

(i) *The modulus of an annulus in Ω separating $\partial\Omega$ is bounded above;*
(ii) *The modulus of a round annulus in Ω separating $\partial\Omega$ is bounded above.*

From the very definition, it is easy to see the equivalence of the uniform perfectness to condition (ii). We denote by $M(\Omega)$ and $M^\circ(\Omega)$ the suprema of annuli and round annuli in Ω separating $\partial\Omega$, respectively. We conventionally define these values to be 0 if such an annulus does not exist. (Similarly, we define the infimum of the empty set to be $+\infty$ in the sequel.) By the standard argument using Teichmüller's extremal ring domains and an modulus estimate for those, we obtain the inequality $M^\circ(\Omega) \le M(\Omega) \le 2M^\circ(\Omega) + C$, where C is an absolute constant (see [31] or [72]). Thus, the equivalence of (i) and (ii) follows.

When Ω is connected, namely, it is a domain, each component of $\widehat{\mathbb{C}} \setminus \Omega$ is simply connected. Hence, by the above characterizations, we observe that the uniform perfectness of $\widehat{\mathbb{C}} \setminus \Omega$ is equivalent to that of $\partial\Omega$. As is well known, $K^{-1}\operatorname{mod} A \le \operatorname{mod} f(A) \le K \operatorname{mod} A$ holds for a K-quasiconformal mapping f (see, for instance, [1]), in particular, in view of condition (i), it turns out that the uniform perfectness is preserved by quasiconformal homeomorphisms.

Note also that, in the first definition of uniform perfectness and condition (ii), the point at infinity plays a special role, while it does not in condition (i).

2.2. **Hyperbolic metric and Fuchsian groups.** In this subsection, we will assume that the set E contains at least three points. Then, the Poincaré-Koebe uniformization theorem tells us that each connected component Ω_0 of Ω has universal covering surface conformally equivalent to the unit disk $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$. In this way, we know of the existence of analytic universal covering map $p = p_{\Omega_0} : \mathbb{D} \to \Omega_0$ of Ω_0 from $\mathbb{D}$. Let $\Gamma = \Gamma_{\Omega_0}$ be the covering transformation group of this covering. Then Γ is a Fuchsian group acting freely on $\mathbb{D}$. Since the hyperbolic (or the Poincaré) metric $\rho_{\mathbb{D}} = |dz|/(1-|z|^2)$ is invariant under the pull-back by elements of Γ, the complete Riemannian metric ρ_{Ω_0} can be defined on Ω_0 by the relation $\rho_{\mathbb{D}} = p^*\rho_{\Omega_0}$, and indeed, this is independent of the particular choice of covering map p. We can define the metric ρ_Ω on Ω component-wise in this way and call it the hyperbolic metric of Ω. In this sense, an open set in $\widehat{\mathbb{C}}$ whose boundary contains at least three points is called hyperbolic. (Caution! The above metric has Gaussian curvature -4, although many authors prefer to use $2\rho_\Omega$ of curvature -1 instead. Therefore, the reader must be careful when comparing other papers.)

The hyperbolic length of a piecewise smooth curve γ in Ω is defined by $\ell_\Omega(\gamma) = \int_\gamma \rho_\Omega(z)|dz|$. A closed curve in Ω is said to be *nontrivial* if it is not homotopic to a point in Ω. We denote by $L(\Omega)$ in the following the infimum $\inf_\gamma \ell_\Omega(\gamma)$ of the hyperbolic lengths of all nontrivial, piecewise smooth, closed curves in Ω. This quantity is 0 if Ω has a puncture, and otherwise, equals the infimum of the hyperbolic lengths of all (simple) closed geodesics in Ω. The hyperbolic distance $d_\Omega(z,w)$ between two points $z,\, w \in \Omega$ is defined as the infimum of the hyperbolic lengths of piecewise smooth curves joining z and w in Ω. Note that the distance is $+\infty$ if z and w belong to different components of Ω. Let $D_\Omega(z_0,r)$ denote the metric ball $\{z\in\Omega; d_\Omega(z_0,z)<r\}$ centered at $z_0\in\Omega$ with radius r. The supremum of those radii $r>0$ for which $D_\Omega(z_0,r)$ is homeomorphic to the usual disk will be called the injectivity radius of Ω at z_0, and denoted by $\iota_\Omega(z_0)$. The infimum of $\iota_\Omega(z_0)$ taken over all $z_0\in\Omega$ will be called the injectivity radius of Ω and denoted by $I(\Omega)$. Then, a simple observation gives us the relation $L(\Omega)=2I(\Omega)$. We end this subsection with the remark that the terminology and notation here can be applied to general hyperbolic Riemann surfaces.

2.3. **Characterization in terms of hyperbolic geometry.** Using the terminology defined in the previous subsection, the uniform perfectness can be characterized as follows.

Theorem 2.2 ([61], [72]). *Let Ω be a hyperbolic open set in $\widehat{\mathbb{C}}$. Then the uniform perfectness of the set $\widehat{\mathbb{C}}\setminus\Omega$ is equivalent to each of the following:*

(i) *The infimum $L(\Omega)$ of hyperbolic lengths of nontrivial closed curves is positive;*
(ii) *The injectivity radius $I(\Omega)$ of Ω is positive;*
(iii) $\inf\{\mathrm{tr}^2 g;\ g\in\Gamma_{\Omega_0}\setminus\{\mathrm{id}\},\ \Omega_0 \text{ is a component of } \Omega\}>4.$

Here, $\mathrm{tr}^2 g$ *denotes the squared trace of an element of* $\mathrm{SL}(2,\mathbb{C})$ *which represents the Möbius transformation g.*

In fact, by a simple computation, we can confirm that the hyperbolic length l_g of the closed geodesic which is represented by a primitive element g in Γ_{Ω_0} is given by the formula $|\mathrm{tr} g| = 2\cosh l_g$. (Note that $\mathrm{tr}^2 g\ge 0$ whenever $g\in\Gamma<\mathrm{Aut}\,(\mathbb{D})$.) Therefore, it is easy to see the three conditions above are equivalent. The remainder is the equivalence of the uniform perfectness of the complement to one of the three conditions. This can be seen from the following comparison theorem between the hyperbolic length $\ell_\Omega[\gamma] = \inf\{\ell_\Omega(\gamma');\gamma'\in[\gamma]\}$ and the extremal length $E_\Omega[\gamma]$ (see, for instance, [1] for definition) of the free homotopy class $[\gamma]$ of a closed curve γ in Ω.

Lemma 2.3 ([72]).

$$\frac{2}{\pi}\ell_\Omega[\gamma] \le E_\Omega[\gamma] \le \frac{\ell_\Omega[\gamma]}{\arctan(1/\sinh \ell_\Omega[\gamma])} \le \frac{2}{\pi}\ell_\Omega[\gamma]e^{\ell_\Omega[\gamma]}.$$

We remark that a slightly weaker form was obtained earlier by Maskit [47].

This estimate can be shown by the standard argument using the collar lemma. Roughly speaking, if there is a closed geodesic of hyperbolic length small enough, then we can take an annulus of sufficiently large modulus as a tubular neighborhood, and vice versa (the converse part is easy to see). The collar lemma describes the above fact quantitatively (see [27], [11]).

Indeed, for $[\gamma]$, we can take the annulus A in Ω called the characteristic ring domain whose core curve is homotopic to γ to express the extremal length concretely by $E_\Omega[\gamma] =$

$2\pi/\mathrm{mod}\, A$ (see Jenkins-Suita [36]. See also [20]). In particular, the inequality $L(\Omega) \le \pi^2/M(\Omega) \le L(\Omega)e^{L(\Omega)}$ follows. It is now immediate to derive the required equivalence.

2.4. **Characterizations in terms of metrics.** In the previous subsection, we saw that the uniform perfectness of the boundary can be characterized in terms of the hyperbolic geometry. Here, we will see that it can also be characterized in terms of the boundary behavior of the hyperbolic density $\rho_\Omega(z)$. Let us first consider the Euclidean distance $\delta_\Omega(z) = \inf\{|z-a|; a \in \partial\Omega\}$ from $z \in \Omega$ to the boundary $\partial\Omega$. Then, the principle of domain extension for the hyperbolic density (see [56]) yields the inequality $\rho_\Omega \le \delta_\Omega$. The continuous metric $|dz|/\delta_\Omega(z)$ is sometimes called the *quasihyperbolic metric* of Ω. The quasihyperbolic metric is widely used for general domains in the case of non-smooth boundary and even in the higher dimensional case when the domain might not have the hyperbolic metric because of its easy access to computations or estimates. See, for instance, [22], [21], [66], [40], [25]. Also in [38] there is a good survey on the quasihyperbolic metric. By virtue of the Koebe one-quarter theorem, we obtain the reverse inequality $1/4\delta_\Omega \le \rho_\Omega$ in the case when Ω is simply connected, nevertheless the quasihyperbolic metric is not necessarily comparable with the hyperbolic metric for general domains. In fact, the comparability gives a characterization of the uniform perfectness of the boundary. Note that we should be careful with the treatment of the point at infinity because the distance function δ_Ω was defined by Euclidean distance. To this end, we introduce the notation $\Omega_r = \{z \in \Omega; \delta_\Omega(z) < r\}$ for $0 < r \le +\infty$.

Theorem 2.4 (Beardon-Pommerenke [8]). *Set $d = d(\partial\Omega)$ for a hyperbolic open set Ω in $\widehat{\mathbb{C}}$. The complement $\widehat{\mathbb{C}} \setminus \Omega$ is uniformly perfect if and only if there exists a constant $c > 0$ such that the inequality $c/\delta_\Omega \le \rho_\Omega$ holds in Ω_d.*

Actually, this theorem seems to be a starting point of the notion of uniform perfectness. The reader may also be referred to [61], [72] and [70]. For the proof, it is essential to make use of the classical Landau theorem, which can be stated as $\rho_{\mathbb{C}\setminus\{0,1\}}(z) \ge 1/|z|(2|\log|z|| + C)$ holds for some constant $C > 0$ in terms of the hyperbolic density. The best constant can be found in [30] and [35]. For further estimates and the relations with other geometric quantities, see also [86] or [72].

2.5. **Schwarzian derivative and universal cover.** The pre-Schwarzian and Schwarzian derivatives of a non-constant meromorphic function f is defined by

$$T_f = \frac{f''}{f'} = (\log f')' \quad \text{and} \quad S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = (T_f)' - \frac{1}{2}(T_f)^2,$$

respectively. For a meromorphic function φ in the hyperbolic open set Ω, we set $\|\varphi\|_{j,\Omega} = \sup_{z\in\Omega} \rho_\Omega(z)^{-j}|\varphi(z)|$. In the case when $j = 2$, this coincides with the hyperbolic sup-norm (see § 4.2) of a holomorphic quadratic differential and plays an important role in the theory of Teichmüller spaces (see [41] or [20]).

Theorem 2.5 (Pommerenke [61], [62]). *Let Ω be a hyperbolic open set. Then the complement $\widehat{\mathbb{C}} \setminus \Omega$ is uniformly perfect if and only if there exists a constant C such that $\|S_p\|_{2,\mathbb{D}} \le C$ holds for any component Ω_0 of Ω and for any analytic universal covering map $p : \mathbb{D} \to \Omega_0$. If $\Omega \subset \mathbb{C}$ in addition, this is equivalent to that there exists a constant C' such that $\|T_p\|_{1,\mathbb{D}} \le C'$ holds for all such p as above.*

The well-known Koebe area theorem in univalent function theory implies the inequalities $|T_f(0)| \le 4$ and $|S_f(0)| \le 6$ for a univalent function $f : \mathbb{D} \to \mathbb{C}$ (see [41]). Let $r_0 = \tanh(I(\Omega)) > 0$. We now assume that $\infty \notin \Omega$ and fix $z_0 \in \mathbb{D}$. Then the function $f(z) = p((r_0 z + z_0)/(1 + \overline{z_0} r_0 z))$ is univalent in $\mathbb{D}$, and thus, satisfies the above inequalities. From this observation, we can deduce the necessity of the above conditions. The sufficiency can also be derived from the same technique as above and from the following two results: $\|S_f\|_{2,\mathbb{D}} \le 2$ *implies the univalence of* f (Nehari [55]) and $\|T_f\|_{1,\mathbb{D}} \le 1$ *implies the univalence of* f (Becker [9]). For related results, see also [53], [44], [72] and [86].

It is well known that $\|S_f\|_{2,\mathbb{D}} \le 6$ and $\|T_f\|_{1,\mathbb{D}} \le 6$ hold for a univalent function $f : \mathbb{D} \to \mathbb{C}$ and it is shown by Beardon-Gehring [7] that $\|S_f\|_{2,\Omega} \le 12$ holds for a univalent function $f : \Omega \to \widehat{\mathbb{C}}$. However, the counterpart of the latter result for the pre-Schwarzian derivative is no longer valid, and moreover, the validity can be used for a characterization of the uniform perfectness.

Theorem 2.6 (Osgood [60]). *Let $\Omega \subset \mathbb{C}$ be a hyperbolic open set. Then, the complement $\widehat{\mathbb{C}} \setminus \Omega$ is uniformly perfect if and only if there exists a constant $C > 0$ such that $\|T_f\|_{1,\Omega} \le C$ holds for an arbitrary univalent holomorphic function f on Ω.*

2.6. **Extension to hyperbolic Riemann surfaces.** The quantities $M(\Omega), L(\Omega)$ and $I(\Omega)$ which were introduced in §§ 2.1–2 can be defined for general hyperbolic Riemann surfaces Ω in the same way. Note, however, that we cannot view the boundary of the surface in general, and hence, we have to interpret the sentence that an annulus $A \subset \Omega$ "separates the boundary" as follows: the inclusion mapping $A \hookrightarrow \Omega$ induces the injective homomorphism $\pi_1(A, *) \to \pi_1(\Omega, *)$ between the fundamental groups. The quasihyperbolic metric $|dz|/\delta_\Omega(z)$ is also difficult to define in the general case because we have no idea to measure the distance from a given point to the "boundary". We have, however, a good substitute $\hat{\rho}_\Omega$, called the Hahn metric, for the quasihyperbolic metric:

$$\hat{\rho}_\Omega(z) = \inf\{\rho_D(z); D \subset \Omega \text{ simply connected and } \ z \in D\},$$

where z is a local parameter of Ω, and indeed, this continuous metric is comparable with the quasihyperbolic metric in the case when Ω is a plane domain (see [26], [24], [52]).

In fact, the uniform perfectness conditions involved with those quantities are still equivalent in this general context.

Theorem 2.7 ([72]). *For any hyperbolic Riemann surface R, the following conditions are equivalent:*

1. $M(R) < +\infty$,
2. $L(R) = 2I(R) > 0$, *and*
3. *The Hahn metric $\hat{\rho}_R$ is comparable with the hyperbolic metric ρ_R.*

Those Riemann surfaces which satisfy one (and hence, all) of the above three conditions may be called *of bounded geometry* (cf. [64]) since they have positive injectivity radii. At present, however, there seems to be no definite terminology for this condition.

3. Analytic characterizations – front view of uniform perfectness –

In the previous section, we considered characterizations of uniform perfectness in terms of geometric properties of the complement. In this section, we try to characterize the uniform perfectness of E in terms of the properties of E itself. Of course, E does not necessarily have a good structure such as complex structures or Riemannian metrics, and therefore, it usually requires more delicate analysis. In what follows, $B(a,r)$ denotes the closed disk centered at a with radius r, and $\mathbb{D}(a,r)$ denotes the open disk of the same data.

3.1. **Characterizations in terms of Hausdorff contents.** Let $h:(0,+\infty)\to(0,+\infty)$ be a non-decreasing continuous function satisfying $\lim_{t\to 0} h(t)=0$. We denote by $\mathcal{L}_h(E)$ the *Hausdorff h-content* of a compact set E in $\mathbb{C}$, which is defined as the infimum of the sum $\sum_k h(d(B_k))$ taken over all countable covers of E by sets B_k. Also, the limit of the infimum of the same sum taken over countable cover B_k satisfying $d(B_k)<\varepsilon$ only as $\varepsilon\to 0$ is known as the *Hausdorff h-measure*, which will be denoted by $\mathcal{H}_h(E)$. Especially when $h(t)=t^\alpha$, we write $\mathcal{L}^\alpha(E)=\mathcal{L}_h(E)$ and $\mathcal{H}^\alpha(E)=\mathcal{H}_h(E)$. As is clear from the definition, $\mathcal{L}_h(E)\le\mathcal{H}_h(E)$, and therefore, $\mathcal{H}_h(E)=0$ implies $\mathcal{L}_h(E)=0$. In turn, the converse of the latter is also true, in other words, $\mathcal{L}_h(E)=0\Leftrightarrow\mathcal{H}_h(E)=0$. The infimum of α with $\mathcal{H}^\alpha(E)=0$ is widely known as the Hausdorff dimension of E, which will be denoted by $\dim E$. By definition, for a linear mapping $f(z)=sz+t$, we can see that $\mathcal{L}^\alpha(f(E))=|s|^\alpha\mathcal{L}^\alpha(E)$ and so on. A characterization of the uniform perfectness is given by Järvi-Vuorinen [34] in terms of Hausdorff contents. As a direct consequence, we can see that a uniformly perfect set has positive Hausdorff dimension. For a more concrete estimate, see also [72] or the proof below.

Theorem 3.1 (Järvi-Vuorinen [34]). *A compact set $E\subset\widehat{\mathbb{C}}$ is uniformly perfect if and only if there exist constants $C>0$ and $\alpha>0$ such that*

$$\mathcal{L}^\alpha(E\cap B(a,r))\ge Cr^\alpha \quad \text{for all} \quad a\in E\setminus\{\infty\} \text{ and } 0<r<d(E)/2. \tag{3.1}$$

3.2. **Proof of Theorem 3.1.** We follow [72] below. Let $\Omega=\widehat{\mathbb{C}}\setminus E$. Noting the trivial inequality $\mathcal{L}^\alpha(B(a,r))\le(2r)^\alpha$, we get the sufficiency immediately. We now show the necessity. Let $0<\alpha<\alpha_0:=\log 2/\log(2e^{M^\circ(\Omega)}+1)$ and take $\beta>1$ so that $\alpha=\log 2/\log(2\beta+1)$. Then we see $\log\beta>M^\circ(\Omega)$. Set $c=1/(2\beta+1)$.

Assuming that $a=0$ and $r=1$, it suffices to estimate $\mathcal{L}^\alpha(E\cap B)$ by a positive constant from below, where $B=B(0,1)$. Let $B_1=B(0,c)$. By assumption, we see $d(E)>2r=2$, and hence $E\setminus B\ne\emptyset$. Since the annulus $A=\{2c<|z|<2\beta c\}$ intersects E by the choice of β, we can take a point $x\in E\cap A$. Letting $B_2=B(x,c)$, we observe that $B_1\cap B_2=\emptyset$ and that $B_2\subset B$. In the same way, we can select disjoint two closed disks $B_{j,1}$ and $B_{j,2}$ inside B_j of radius c^2. Continuing this procedure, for each $(j_1,\dots,j_k)\in\{1,2\}^k$, we can choose a closed disk $B_{j_1,\dots,j_k}$ of radius c^k so that the following conditions are satisfied: (1) the center of $B_{j_1,\dots,j_k}$ is contained in E, (2) $B_{j_1,\dots,j_k}\subset B_{j_1,\dots,j_{k-1}}$ and (3) $B_{j_1,\dots,j_{k-1},1}\cap B_{j_1,\dots,j_{k-1},2}=\emptyset$. Then

$$F=\bigcap_{k=1}^{\infty}\bigcup_{j_1,\dots,j_k\in\{1,2\}} B_{j_1,\dots,j_k}$$

is a Cantor set contained in E. In spite of the superfluousness for the proof, it may be interesting to note that the family $(B_{j_1,\dots,j_k})$ for a fixed k is a finite covering of the set F and $\sum d(B_{j_1,\dots,j_k})^\alpha=2^k(2c^k)^\alpha=2^\alpha$, and thus, $\mathcal{H}^\alpha(F)\le 2^\alpha$. We now give a lower estimate of $\mathcal{L}^\alpha(F)$. Let μ be the measure which is induced by the Bernoulli measure of the equidistribution on $\{1,2\}^{\mathbb{N}}$. Namely, μ is a Borel probability measure with support F such that $\mu(B_{j_1,\dots,j_k}) = 2^{-k}$. We now show that $\mu(A) \le 18t^\alpha$ for any closed disk $A = B(b,t)$. We may assume that $t < 1$ because otherwise this trivially holds. Choose k so that $c^{k+1} < t \le c^k$. If we set $J = \{j = (j_1,\dots,j_k); A \cap B_j \neq \emptyset\}$, then $\cup_{j\in J} B_j \subset B(b, t+2c^k)$. Taking the area of disks into account, we obtain $\#J \cdot (c^k)^2 \le (t+2c^k)^2$, and hence, $\#J \le (2+tc^{-k})^2 \le 9$. Using the last inequality, we compute $\mu(A) \le \mu(\cup_{j\in J} B_j) = \#J 2^{-k} \le 18 \cdot 2^{-k-1} < 18t^\alpha$. Now the desired inequality has been shown.

We are now able to get a lower estimate of $\mathcal{L}^\alpha(F)$ as follows. Let (A_n) be an arbitrary countable cover of F. Setting $d_n = d(A_n)$, we can see that each A_n is contained in a closed disk A_n' of radius d_n. Therefore, the above inequality can be applied to conclude $\mu(A_n') \le 18{d_n}^\alpha$. From this, we see that

$$1 = \mu(F) \le \sum_n \mu(A_n) \le 18 \sum_n {d_n}^\alpha,$$

and, taking the infimum over all possible countable cover of F, we finally get $\mathcal{L}^\alpha(F) \ge 1/18$. Therefore, we have obtained the inequality $\mathcal{L}^\alpha(E \cap B) \ge 1/18$ for any $\alpha < \alpha_0$. Now we can use the left continuity of α-dimensional Hausdorff content with respect to the dimension α (see [59]) to get $\mathcal{L}^{\alpha_0}(E \cap B) \ge 1/18$. In particular, the concrete estimate $\dim E \ge \alpha_0 = \log 2/\log(2e^{M^\circ(\Omega)}+1) \ge \log 2/(M^\circ(\Omega) + \log 3)$ follows.

3.3. **Characterization in terms of logarithmic capacity.** The logarithmic capacity $\operatorname{Cap} E$ of a compact set E in $\mathbb{C}$ can be defined as the number so that Green's function G of the (unique) unbounded component Ω_∞ of $\widehat{\mathbb{C}} \setminus E$ with pole at infinity has the asymptotic behavior $G(z) = \log|z| - \log \operatorname{Cap} E + o(1)$ as $z \to \infty$. Here, we define $\operatorname{Cap} E$ to be 0 if Ω_∞ carries no Green's function. For instance, $\operatorname{Cap} B(a,r) = r$, and the relation $\operatorname{Cap} f(E) = |s| \operatorname{Cap} E$ holds for the linear mapping $f(z) = sz + t$. Pommerenke has given the following remarkable characterization of the uniform perfectness.

Theorem 3.2 (Pommerenke [61]). *A compact set $E \subset \widehat{\mathbb{C}}$ is uniformly perfect if and only if there exists a constant $c > 0$ with the property that*

$$\operatorname{Cap}(E \cap B(a,r)) \ge cr, \quad \textit{for all} \quad a \in E \setminus \{\infty\} \textit{ and } 0 < r < d(E).$$

Using the characterization of capacity as the transfinite diameter, Pommerenke has reached this result by ingenious computations. We will give another proof here by employing the estimate for Hausdorff contents in § 3.1. If we take a closer look at the proof due to Tsuji [80, pp. 65–66] for Frostman's theorem: *The condition $\mathcal{H}_h(E) > 0$ for some h with $\int_0^1 h(t)dt/t < +\infty$ implies* $\operatorname{Cap} E > 0$, we can read the following concrete estimate:

$$\operatorname{Cap} E \ge \exp\left(-C_1 \frac{\int_0^1 h(t)dt/t}{\mathcal{L}_h(E)} \right)$$

for a compact set E contained in a closed disk of diameter 1, where $C_1 > 0$ is an absolute constant. We now assume (3.1). Letting $f(z) = z/2r$, we apply the above estimate to the set $F = f(E \cap B(a,r))$ and the measure function $h(t) = t^\alpha$ to obtain $\operatorname{Cap}(E \cap B(a,r)) = 2r\operatorname{Cap}(F) \ge 2r\exp(-2^\alpha C_1/\alpha C)$ because (3.1) implies $\mathcal{L}^\alpha(F) = (2r)^{-\alpha}\mathcal{L}^\alpha(E \cap B(a,r)) \ge 2^{-\alpha}C$. The desired estimate now follows with $c = 2\exp(-2^\alpha C_1/\alpha C)$.

By the above result, the lower capacity density $\liminf_{r\to 0} \mathrm{Cap}\,(E \cap B(a,r))/r$ can be estimated from below. In particular, the Wiener criterion can be applied to conclude that each point of a uniformly perfect set is regular with respect to the Dirichlet problem (cf. [80]). Indeed, as we shall see in the next subsection, a stronger regularity holds for uniformly perfect sets.

Note that there are characterizations in terms of condenser capacities (see [34] or [65]).

3.4. **Hölder regularity in the Dirichlet problem.** Let Ω be an open set with boundary of positive capacity and consider a solution u of the Dirichlet problem

$$(\partial_x^2 + \partial_y^2)u = 0 \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega$$

in the sense of Perron-Wiener-Brelot. For simplicity, we consider only a bounded solution u for a bounded Borel function φ. Then there exists a unique solution u for φ, which will be denoted by $u = H^\Omega\varphi$. For the details, the reader is referred to, for instance, [80], [29] and [16].

The harmonic measure is frequently used to measure the size of sets in the boundary of a domain (see [56]). Let χ_F be the defining function for a Borel set F in $\partial\Omega$. Then the solution $H^\Omega\chi_F$ of the Dirichlet problem in Ω with the boundary data χ_F is called the harmonic measure of F relative to Ω and denoted by $\omega(\cdot, F, \Omega)$. For a fixed $z \in \Omega$, the set function $F \mapsto \omega(z, F, \Omega)$ can be regarded as a Radon measure representing the bounded linear functional $\varphi \mapsto H^\Omega\varphi(z)$ on the Banach space consisting of bounded continuous functions on $\partial\Omega$. Let E be a compact set in $\widehat{\mathbb{C}}$ and let $\Omega = \widehat{\mathbb{C}} \setminus E$. For each $a \in \partial_{\mathrm{b}}\Omega$ and each $0 < r$ we denote by $\omega_{a,r,\Omega}$ the harmonic measure of $\Omega \cap \partial B(a,r)$ relative to $\Omega \cap \mathbb{D}$ and denote by $\hat{\omega}_{a,r,\Omega}$ the harmonic measure of $\partial\Omega \setminus \mathbb{D}(a,r)$ relative to Ω. These are called the local and global harmonic measures for Ω at a, respectively (see [70]). The minimum principle implies that $\hat{\omega}_{a,r,\Omega} \le \omega_{a,r,\Omega}$ in $\Omega \cap \mathbb{D}(a,r)$. Then the following result holds.

Theorem 3.3. *For a number $0 < \alpha < 1$ we consider the following conditions:*

(i) *There exists a positive constant C such that $\omega_{a,r,\Omega}(z) \le C(|z-a|/r)^\alpha$ holds for any $a \in \partial_{\mathrm{b}}\Omega$, $0 < r < d(\partial\Omega)$ and $z \in \Omega \cap \mathbb{D}(a,r)$;*

(ii) *There exists a positive constant C such that $\hat{\omega}_{a,r,\Omega}(z) \le C(|z-a|/r)^\alpha$ holds for any $a \in \partial_{\mathrm{b}}\Omega$, $0 < r < d(\partial\Omega)$ and $z \in \Omega \cap \mathbb{D}(a,r)$;*

(iii) *For a boundary function φ on $\partial_{\mathrm{b}}\Omega$ which is Hölder continuous with exponent α, the solution $H^\Omega\varphi$ is Hölder continuous with the same exponent, too.*

Then, E is uniformly perfect if and only if (i) *is valid for some α and the latter implies* (ii) *and* (iii). *If, in addition, Ω is a bounded domain,* (i) *is equivalent to* (ii) *and, conversely, if* (iii) *is valid for some $\alpha' > \alpha$, then* (i) *is valid for α.*

The equivalence of the uniform perfectness to property (i) is essentially due to Ancona [4], however, he treats only the case of $n > 2$ there. The statement in the above form can be found in [70]. We use Theorem 3.2 for the proof. Also the implications (i) $\Rightarrow$ (ii), (iii) are due to [70]. The latter part of the theorem is very recently proved by Aikawa [2].

From the above result, in particular, it can be seen that the Green's function on a domain with uniformly perfect boundary is Hölder continuous near the boundary. This observation was stated in textbook [15] without proof. The probably first proof for this fact appeared in [42] in print. We also find a proof in [65]. We remark that the converse is not true (see § 3.6). As is noted in [15], from the theorem of Carleson [14] that a compact set of Hausdorff dimension less than α is removable for Hölder continuous harmonic functions with exponent α, we can deduce that the boundary has Hausdorff dimension at least α under the assumption of condition (i) in Theorem 3.3.

3.5. **Characterizations in terms of uniformizing Fuchsian groups.** This subject might be better to be included in the previous section. We have, however, decided to discuss it here because we need an estimate of logarithmic capacity for the proof. Let Ω be a hyperbolic domain in $\widehat{\mathbb{C}}$ and let Γ be the Fuchsian group which is a covering transformation group of an analytic universal covering map $p : \mathbb{D} \to \Omega$ of Ω. We then have the following characterizations.

Theorem 3.4 (Pommerenke [62]). *The boundary $\partial\Omega$ is uniformly perfect if and only if each of the following conditions holds:*

(i) *There exists a constant $c_1 > 0$ such that*

$$\left| \frac{g(\zeta) - \zeta}{1 - \bar{\zeta} g(\zeta)} \right| \geq c_1 \quad \text{for all} \quad \zeta \in \mathbb{D} \text{ and } g \in \Gamma \setminus \{\mathrm{id}\};$$

(ii) *There exists a constant $c_2 > 0$ such that*

$$\prod_{g \in \Gamma \setminus \{\mathrm{id}\}} \left| \frac{g(\zeta) - \zeta}{1 - \bar{\zeta} g(\zeta)} \right| \geq c_2 \quad \text{for all} \quad \zeta \in \mathbb{D}.$$

From condition (ii), we see that the orbit $\{g(\zeta);\, g \in \Gamma\}$ of a point $\zeta \in \mathbb{D}$ under Γ is an interpolating sequence with respect to the bounded analytic functions on the unit disk (see [13]). Condition (i) above is merely rephrasing Theorem 2.2, while condition (ii) is apparently much more stronger than (i). For the proof, we may assume that $\infty \in E = \widehat{\mathbb{C}} \setminus \Omega$. From Theorem 3.2, it follows that for a constant $c > 0$ the inequality $\mathrm{Cap}\,(E \cap B(a, r)) \geq cr$ holds for all $a \in E \setminus \{\infty\}$ and $r > 0$. Fix $\zeta_0 \in \mathbb{D}$. By replacing $p(\zeta)$ by $p((\zeta + \zeta_0)/(1 + \overline{\zeta_0}\zeta))$ if necessary, we may assume $\zeta_0 = 0$. Set $E^* = \{(a - z_0)^{-1};\, a \in E\}$, where $z_0 = p(0)$. Take a point $a_0 \in \partial\Omega$ so that $\delta_\Omega(z_0) = |z_0 - a_0| =: r_0$. Then, we see

$$\begin{aligned} \mathrm{Cap}\, E^* &\geq \mathrm{Cap}\,(E \cap B(z_0, 2r_0))^* \geq \mathrm{Cap}\,(E \cap B(z_0, 2r_0))/4{r_0}^2 \\ &\geq \mathrm{Cap}\,(E \cap B(a_0, r_0))/4{r_0}^2 \geq c/4r_0. \end{aligned}$$

Noting that $r_0 \leq 1/\rho_\Omega(z_0) = |p'(0)|$, we finally get $\mathrm{Cap}\, E^* \geq c/4|p'(0)|$. We now let G be Green's function of Ω with pole at z_0. Then $G(z) = \log(1/|z - z_0|) - \log \mathrm{Cap}\, E^* + o(1)$ as $z \to z_0$. On the other hand, Myrberg's theorem [80, p.522] implies that

$$\begin{aligned} G(p(\zeta)) &= \sum_{g \in \Gamma} \log \left| \frac{1 - \overline{g(0)}\zeta}{\zeta - g(0)} \right| \\ &= \log \frac{1}{|\zeta|} + \sum_{g \in \Gamma \setminus \{\mathrm{id}\}} \log \frac{1}{|g(0)|} + O(|\zeta|), \end{aligned}$$

and hence, $\prod_{g \in \Gamma \setminus \{\mathrm{id}\}} |g(0)| = |p'(0)| \mathrm{Cap}\, E^* \geq c/4$. Now the proof is complete.

We remark that other characterizations of the uniform perfectness in terms of Green's function or Fuchsian groups are given by González [23].

3.6. **Markov inequalities.** There is a series of results, usually called the Markov inequalities, which are playing an important role in approximation theory. It was originally utilized for the approximation of a real function on the interval $[0, 1]$, however, it have been extended in various directions and found many applications. We give some of those formulations here. A compact set E in the complex plane is said to *preserve the global Markov inequality* if there exist constants $C > 0$ and $\rho \geq 1$ such that $\|P'\|_E \leq C(\deg P)^\rho \|P\|_E$ holds for any polynomial P (of a complex variable). Here $\|f\|_E = \sup\{|f(z)|; z \in E\}$. A compact set E in $\widehat{\mathbb{C}}$ is said to *preserve the local Markov inequality* if for each positive integer k there exists a constant $c = c(k)$ such that $r\|P'\|_{E\cap B(a,r)} \leq c\|P\|_{E\cap B(a,r)}$ holds for all $a \in E$, $0 < r < d(E)$ and polynomial P of degree k in z. Then the following can be shown.

Theorem 3.5 (Lithner [42]). *A compact set $E \subset \widehat{\mathbb{C}}$ is uniformly perfect if and only if E preserves the local Markov inequality. Furthermore, if $E \subset \mathbb{C}$ in addition, then E preserves the global Markov inequality.*

It is known that if Green's function of a domain with pole at infinity is Hölder continuous near the boundary then the global Markov inequality is preserved. Lithner [42] gave a non-uniformly perfect Cantor set whose complement carries Hölder continuous Green's function. In particular, we see that the preservation of the global Markov inequality does not necessarily imply the preservation of the local one. Note that a similar domain to Lithner's, which has, however, no degenerate boundary components, can be found in [70].

4. Remarkable properties of uniformly perfect sets

In this section, we collect some of remarkable properties of uniform perfectness, which does not necessarily give characterizations of it.

4.1. **The bottom of spectrum.** In this subsection, Ω means a hyperbolic domain. We recall the fact that $\partial\Omega$ is uniformly perfect if and only if the infimum of the hyperbolic lengths of simple closed geodesics of Ω is positive. On the other hand, through the Selberg trace formula, it is recognized that there is a strong analogy between the length spectrum of simple closed geodesics and the spectrum of the Laplacian on Ω (see, for instance, [11]). We will explain that, in our context, there is a result in this direction.

Let $-\Delta$ be the Laplace-Beltrami operator on Ω with respect to the hyperbolic metric, namely, $-\Delta = -\rho_\Omega(z)^{-2}(\partial_x^2 + \partial_y^2)$. Since this acts on the space $C_c^\infty(\Omega)$ as a non-negative self-adjoint operator, $-\Delta$ can be uniquely extended to a(n unbounded) non-negative self-adjoint operator on $L^2(\Omega)$ and its spectrum is contained in $[0, +\infty)$. The infimum of the spectrum will be called the *bottom* of the spectrum of Ω and will be denoted by $\lambda(\Omega)$. This quantity is known to connect with other global geometric quantities (see [77]). In particular, if we denote by $\eta(\Omega)$ the critical exponent of the Fuchsian group Γ_Ω uniformizing Ω, which is also known to equal the Hausdorff dimension of the conical limit set of Γ_Ω (see [57]), then by the Elstrodt-Patterson-Sullivan theorem [77] we obtain the relation

$$\lambda(\Omega) = \begin{cases} 1, & \text{if } 0 \leq \eta(\Omega) \leq \frac{1}{2}, \\ 4\eta(\Omega)(1 - \eta(\Omega)), & \text{if } \frac{1}{2} \leq \eta(\Omega) \leq 1. \end{cases}$$

Note, in particular, that $\lambda(\Omega) > 0 \Leftrightarrow \eta(\Omega) < 1$. Fernández showed the following result (see also [19] and [3] for related topics).

Theorem 4.1 (Fernández [18]). *The bottom $\lambda(\Omega)$ of the spectrum of Ω is positive for a domain with uniformly perfect boundary.*

By using Cheeger's inequality, the quantitative result $\lambda(\Omega) \geq 1/(1+\pi/L(\Omega))^2$ has been obtained [71]. As is discussed in [19], the converse is not true in general.

4.2. **Holomorphic quadratic differentials.** We will give a brief exposition of the relation between the uniform perfectness and the Bers spaces, which are most important Banach spaces in connection with the Teichmüller spaces. For a holomorphic quadratic differential $\varphi = \varphi(z)dz^2$ on a hyperbolic domain Ω we define the two kinds of norm $\|\varphi\|_1 = \iint_\Omega |\varphi(z)|dxdy$, $\|\varphi\|_\infty = \sup_{z\in\Omega} \rho_\Omega(z)^{-2}|\varphi(z)|$ and we denote by $A_2(\Omega)$ and $B_2(\Omega)$ the corresponding complex Banach spaces. The norm $\|\varphi\|_\infty$ is same as $\|\varphi\|_{2,\Omega}$ defined in § 2.5 and called the hyperbolic norm or the Nehari norm. Whether the inclusion relation $A_2(\Omega) \subset B_2(\Omega)$ holds had been a long-standing problem, and was completely solved by Niebur-Sheingorn [58]. This inclusion relation holds if and only if $L^*(\Omega) = \inf \ell_\Omega(\gamma)$ is positive, where the infimum is taken over all nontrivial, piecewise smooth, closed curves γ in Ω which are not loops winding about one puncture. A more quantitative result can be found in [49] or [69]. Since $L(\Omega) = L^*(\Omega)$ holds if Ω has no punctures, we, in particular, obtain the following.

Theorem 4.2. *Let Ω be a hyperbolic plane domain without punctures. Then $A_2(\Omega) \subset B_2(\Omega)$ holds if and only if $\partial\Omega$ is uniformly perfect.*

In terms of holomorphic quadratic differentials, the author introduced the conformally invariant metric $q_\Omega(z) = \sup\{|\varphi(z)|^{1/2}; \varphi \in A_2(\Omega), \|\varphi\|_1 = \pi\}$ in [73] and [69]. Using this metric, the following characterizations of the uniform perfectness can be obtained. These characterizations are also extended to general hyperbolic Riemann surfaces and actually are equivalent to be of bounded geometry.

Theorem 4.3 ([69]). *For a hyperbolic domain Ω, the boundary is uniformly perfect if and only if one of the following conditions holds:*

(i) *The metric q_Ω is comparable with the Hahn metric $\hat{\rho}_\Omega$;*
(ii) *The metric q_Ω is comparable with the hyperbolic metric ρ_Ω.*

We may replace the Hahn metric by the quasihyperbolic metric in the case when $\Omega \subset \mathbb{C}$. We note that q_Ω is always majorized by the Hahn metric, whereas there is no natural order relation between q_Ω and ρ_Ω (see [73]).

4.3. **Miscellaneous properties.** Here we give a small collection of references where other characterizations or properties of uniform perfectness can be found. For characterizations in terms of BMO spaces, see [60], [24] and [28]. The idea to characterize the uniform perfectness by the relationship between the hyperbolic distance and the Euclidean distance are due to [61] or [43]. The relation with the quasiconformal homogeneity was pointed out by [46].

5. Examples of uniformly perfect sets

In the preceding sections, we have seen general properties of uniformly perfect sets. In this section, we will survey what kind of sets are actually uniformly perfect through concrete examples.

5.1. **Kleinian groups.** A Kleinian group is a discrete subgroup of the complex Lie group $\mathrm{PSL}(2, \mathbb{C})$. The reader is referred to [48] or [50] for details. The region of discontinuity of a Kleinian group G concerning the action on $\widehat{\mathbb{C}}$ will be denoted by $\Omega(G)$ and the complement, which is called the limit set of G, will be denoted by $\Lambda(G)$. For simplicity, we consider only Kleinian groups without elliptic elements. To avoid the trivial case, we also restrict ourselves to the non-elementary case when the limit set consists of at least three points, and hence, is perfect. Then we can show the following.

Theorem 5.1 ([75]). *If the quotient Riemann surface $R = \Omega(G)/G$ satisfies $L^*(R) > 0$, then the limit set $\Lambda(G)$ is uniformly perfect.*

We use condition (i) in Theorem 2.2. Indeed, let γ be an arbitrary nontrivial closed curve in $\Omega(G)$. Then the projection γ' of γ onto R is also a nontrivial closed curve and is not homotopic to a multiple of a loop winding around a puncture once. (The latter can be seen from the fact that $\Lambda(G)$ is perfect.) Therefore, we compute $\ell_\Omega(\gamma) \geq \ell_R(\gamma') \geq L^*(R)$ and get $L(\Omega(G)) \geq L^*(R) > 0$.

By the theorem of Niebur-Sheingorn stated in § 4.2, we see that the condition $L^*(R) > 0$ is equivalent to $A_2(R) \subset B_2(R)$. The condition $L^*(R) > 0$ always holds for a Riemann surface of finite analytic type, and hence, we conclude that the limit set of a non-elementary, finitely generated, Kleinian group is uniformly perfect by Ahlfors' finiteness theorem.

We note here historical remarks. The above result was proved by [8] in the case of finitely generated Schottky groups, however, in view of the characterization of the uniform perfectness in terms of capacity density, it seems to be implied by a result of Tsuji [79]. The first proof appeared in [62] for the case of general finitely generated Kleinian groups. Canary [12] extended it to the case of analytically finite Kleinian groups. We remark that $\Lambda(G)$ may not be uniformly perfect when G is infinitely generated. A concrete example can be found in [75]. The uniform perfectness of the limit set has been effectively used by, e.g., Bishop-Jones [10] in the theory of Kleinian groups.

5.2. **Complex dynamics.** Let f be a rational function of one variable with $\deg f \geq 2$. Then the Julia set $J(f)$ is invariant under the action of f, and thus, it has a kind of self-similarity. For the fundamentals of complex dynamics such as Julia sets, the reader is referred to, for instance, [6], [15] or [54]. We know the following fact.

Theorem 5.2. *The Julia set of a rational function of degree at least two is uniformly perfect.*

This result was proved around 1992 by Mañe-da Rocha [45], Hinkkanen [32] and Eremenko [17], independently. The case when f is hyperbolic was shown by Pommerenke [62] earlier. Their methods, more or less, relied on the contradiction except for that of Eremenko, so no concrete estimate was given. The author has given an explicit estimate in [74] for uniform perfectness of the Julia set in terms of geometric quantities defined in the Fatou set. (In fact, the idea was same as that of Eremenko, however, his paper [17] had not been submitted, so the author had not been aware of that paper after completion of the manuscript.) The example given by Baker [5] tells us that the Julia set of a transcendental entire function is no longer uniformly perfect in general. Zheng [89], [87] made investigations in the case of transcendental meromorphic functions.

As for the Julia set of a rational semigroup, Hinkkanen-Martin [33] proved the uniform perfectness in the case when each generator has degree at least two, and Stankewitz [68] showed the general case including a finitely generated Kleinian group and a self-similar fractal. Stankewitz [67] investigated also attractors.

6. Prospects

We end this article with possible directions of future investigations.

6.1. **Higher dimensional case.** In this article, we have focused on the two-dimensional case, however, we can consider the same notion in Euclidean space or sphere of general dimension, and even in certain metric spaces. There are several researches in this direction already. Indeed, Ancona [4], Siciak [65] and Aikawa [2] discuss the uniform perfectness in higher dimensional cases from the potential theoretic point of view. Note, however, that the form of Green kernel is essentially different between the cases $n = 2$ and $n \geq 3$, and hence, equivalent notions in two-dimensional case might have different generalizations.

Vuorinen [83] and Järvi-Vuorinen use the characterization of the uniform perfectness in terms of condenser capacity to investigate the boundary behaviour of higher-dimensional quasi-regular mappings.

Tukia-Väisälä [81] treats the case of general metric spaces, however, there are few substantial investigations except for, e.g., Trotsenko-Väisälä [78] in this direction.

We also notice that the uniform perfectness discussed here is, by definition, essentially a one-dimensional concept along the radial direction. Therefore, there is a natural limitation of the estimate of Hausdorff dimension from below. If we have a notion which generalizes the uniform perfectness in this respect, then we would obtain a more effective estimate. As such a notion, we may take a similar concept to the preservation of Markov inequalities. A condition for a set to preserve the local Markov inequality for real polynomials of n variables is known, for instance, by [37]. A more generalized concept has been given by Väisälä-Vuorinen-Wallin [82]. In addition, there is a lower estimate of Hausdorff dimension in terms of Markov inequalities, see [85].

6.2. **Opposite notions.** The notion of uniform perfectness is concerning the density of the set. On the other hand, the notion concerning the coarseness of the set might be useful, too. As such a notion, we may draw the reader's attention to the notion of porosity. This means that there exists a constant $c > 0$ such that for any $a \in E$ and $r > 0$ we can take a ball of the form $B(b, cr)$ in $B(a, r)$ such that $B(b, cr) \cap E = \emptyset$. Then, by using the typical argument of the Lebesgue points, we can immediately conclude that the set has area zero. Moreover, by the aid of packing dimensions, we would get an upper estimate of Hausdorff dimension of the set. Hence, this concept is effectively utilized, for instance, in the theory of complex dynamics (see, e.g., [39], [51] and [63]).

6.3. **Refinement.** The uniform perfectness is invariant under the similarities. If we abandon it and allow c to depend on r in the definition in § 1.1, then we will get a more refined notion. Even in this case, a detailed analysis will enable us still to get useful information about the set. There seems to be quite few researches in this direction so far, however, the author is trying to contribute something [76].

Recall that the uniform perfectness is uniform not only in $r \in (0, d(E))$ but also in $a \in E$. Thus it can be localized for a fixed point $a \in E$. This approach was already used by Vuorinen [84] to investigate angular limits of quasiconformal mappings. Zheng [88] has recently revealed that the localized condition can give good enough results. From the same point of view, the strong regularity is investigated in the Dirichlet problem [70].

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Department of Mathematics, Kyoto University, 606-8502 Kyoto, Japan
Current address: Department of Mathematics, University of Helsinki, P.O.Box 4 (Yliopistonkatu 5), FIN-00014, Helsinki, Finland
E-mail address: `sugawa@kusm.kyoto-u.ac.jp`