GEOMETRIC PROPERTIES OF FUNCTIONS WITH SMALL SCHWARZIAN DERIVATIVES

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Abstract. This note is a summary of the joint work [6] with Ji-A Kim at POSTECH, Korea, with some expository accounts on the Schwarzian derivative.

1. Schwarzian derivative

For a non-constant meromorphic function f in a plane domain, the Schwarzian derivative S_f of f is defined by

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

It is easily seen that S_f is holomorphic at z_0 if and only if f is locally univalent at z_0. The Schwarzian derivative annihilate the effect of Möbius transformations: $S_g = 0$ if and only if g is (a restriction of) a Möbius transformation and, moreover, $S_{L\circ f} = S_f$ for any Möbius transformation L. Therefore, one may think that the Schwarzian derivative measures the deviation of the function from Möbius transformations. As evidence of this heuristic principle, we first point out the following classical result.

Theorem 1.1. *If f is univalent meromorphic in the unit disk $\mathbb{D}$, then*

$$|S_f(z)| \le 6(1-|z|^2)^{-2}.$$

Conversely, if a meromorphic function f in $\mathbb{D}$ satisfies

$$|S_f(z)| \le 2(1-|z|^2)^{-2},$$

then f is univalent in $\mathbb{D}$. The numbers 6 and 2 are sharp.

The former result was first proved by Kraus [7] but had been forgotten for a long time. Nehari [10] re-discovered it and showed the latter result. The Koebe function $K(z) = z/(1-z)^2$ satisfies

$$S_K(z) = \frac{-6}{(1-z^2)^2},$$

which shows the bound 6 is sharp. On the other hand, the function $L(z) = (1/2)\log(1+z)/(1-z)$ which maps $\mathbb{D}$ onto the parallel strip $|\operatorname{Im} w| < \pi/2$ satisfies

$$S_L(z) = \frac{2}{(1-z^2)^2}.$$

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Because the parallel strip is not a quasidisk, this example shows that the bound 2 is sharp in the latter part of the theorem. This reasoning is explained, for instance, in [8]. Hille [5] showed the sharpness of 2 directly by giving the non-univalent example $f(z) = [(1+z)/(1-z)]^{i\varepsilon},\ \varepsilon > 0$.

There are other variations of Nehari's univalence criterion:

Theorem 1.2 (Nehari [10], Pokornyi [12]). *If f satisfies one of the following conditions in $\mathbb{D}$, then f is univalent in $\mathbb{D}$:*

$$|S_f(z)| \le \frac{\pi^2}{2},$$
$$|S_f(z)| \le 4(1-|z|^2)^{-1}.$$

These numbers are sharp.

Extremal functions are given, respectively, by

$$\tan\frac{\pi z}{2} \quad \text{and} \quad \frac{z}{2(1-z^2)} + \frac{1}{4}\log\frac{1+z}{1-z}.$$

Based on these results, more general univalence criteria were deduced by Avkhadiev and *et al* (see [1]).

2. Connection with a linear ODE

For a given holomorphic function φ in the unit disk $\mathbb{D}$, we can construct a locally univalent meromorphic function f so that $S_f = \varphi$ in $\mathbb{D}$. Indeed, let y_0 and y_1 be the analytic solutions to the ODE

$$2y'' + \varphi y = 0$$

in $\mathbb{D}$ with the initial conditions

$$y_0(0) = 1, \quad y_1(0) = 0,$$
$$y_0'(0) = 0, \quad y_1'(0) = 1.$$

Note here that the Wronskian is identically 1 :

$$y_0 y_1' - y_0' y_1 \equiv 1.$$

Then the quotient $f = y_1/y_0$ is a desired one, because the logarithmic derivative of

$$f' = \frac{y_0 y_1' - y_0' y_1}{y_0^2} = \frac{1}{y_0^2}$$

yields

$$\frac{f''}{f'} = -\frac{2y_0'}{y_0}.$$

Hence,

$$\begin{aligned} S_f &= \left(-\frac{2y_0'}{y_0}\right)' - \frac{1}{2}\left(-\frac{2y_0'}{y_0}\right)^2 \\ &= -\left(\frac{2y_0''}{y_0}\right) + 2\left(\frac{y_0'}{y_0}\right)^2 - 2\left(\frac{y_0'}{y_0}\right)^2 \\ &= \varphi. \end{aligned}$$

The constructed function f satisfies $f(0) = 0, f'(0) = 1$ and $f''(0) = 0$. The third condition have been missed by some authors (including myself [13]). This condition is, however, too strong as long as the classical theory of univalent functions is concerned. Therefore, we will introduce special classes of normalized functions.

Let $\mathcal{M}$ be the set of meromorphic functions f in the unit disk $\mathbb{D}$ with $f(0) = 0, f'(0) = 1$. For a complex number c, set

$$\mathcal{M}(c) = \{f \in \mathcal{M} : f''(0) = 2c\}.$$

Note that $f \in \mathcal{M}(c)$ has a series expansion of the form $f(z) = z + cz^2 + \cdots$. Then we can see that *for φ and for $c \in \mathbb{C}$, there is the unique function $f = f_{\varphi,c}$ in $\mathcal{M}(c)$ for which $S_f = \varphi$ holds.* Indeed, such an f can be given by

$$f_{\varphi,c} = \frac{y_1}{y_0 - cy_1} = \frac{f_{\varphi,0}}{1 - cf_{\varphi,0}},$$

where $f_{\varphi,0} = y_1/y_0$.

Set $K(\varphi) = \{c \in \mathbb{C} : 1/c \notin f_{\varphi,0}(\mathbb{D})\}$. Then the set $K(\varphi)$ is always compact. Note that $f_{\varphi,c}$ is pole-free (i.e., analytic) if and only if $c \in K(\varphi)$. It may be interesting to see that $|c| \le 2$ for each $c \in K(\varphi)$ if $f_{\varphi,0}$ is univalent meromorphic. Recall here that the Koebe one-quarter theorem asserts that every omitted value ω of a univalent function $f(z) = z + a_2z^2 + \cdots$ satisfies the inequality $|\omega| \le 1/4$, namely, $c = 1/\omega$ satisfies $|c| \le 4$ and the number 4 is best possible. The constraint $|c| \le 2$ in our situation comes from the special nature $f''(0) = 0$ of f.

3. Weight functions

A function $A(x)$, $0 \le x < 1$, is called a *weight function* if it is locally Lipschitz, non-decreasing, and positive. A typical and important example is given by $A(x) = C(1-x^2)^{-\mu}$, where $C > 0$ and $\mu \ge 0$ are constants.

Let U_0, U_1, V_0 and V_1 be the functions on $[0, 1)$ determined by the initial value problem of the ODE's:

$$\begin{aligned} 2U_0'' &= AU_0, & U_0(0) &= 1, & U_0'(0) &= 0,\\ 2U_1'' &= AU_1, & U_1(0) &= 0, & U_1'(0) &= 1,\\ 2V_0'' &= -AV_0, & V_0(0) &= 1, & V_0'(0) &= 0,\\ 2V_1'' &= -AV_1, & V_1(0) &= 0, & V_1'(0) &= 1. \end{aligned}$$

When we need to indicate the weight function A, we write, for example, $U_0(x, A) = U_0(x)$. Note that $U_0 > 0$ and $U_0' > 0$ hold on the interval $[0, 1)$ for any weight function A.

Nehari [11] established a general univalence criterion:

Theorem 3.1. *Let A be a weight function.*

(i) *If $A(x)(1 - x^2)^2$ is non-increasing in $0 \le x < 1$, and*
(ii) *if $V_0(x, A)$ is positive for $0 \le x < 1$,*

then the condition $|S_f(z)| \le A(|z|)$ for a function $f \in \mathcal{M}$ implies univalence of f in $\mathbb{D}$.

Example 3.2. For $A(x) = \pi^2/2$, one has $V_0(x) = \cos(\pi x/2)$.
For $A(x) = 4(1 - x^2)^{-1}$, one has $V_0(x) = 1 - x^2$.
For $A(x) = 2(1 - x^2)^{-2}$, one has $V_0(x) = \sqrt{1 - x^2}$.

In this note we consider the problem: what geometric properties can we say about those functions with prescribed growth of the Schwarzian derivatives?

We remind the reader of basic terminology in univalent function theory (see [3] for a conprehensive treatment). A function $f \in \mathcal{M}$ is called *starlike* if f is univalent analytic and the image $f(\mathbb{D})$ is starlike with respect to the origin, in other words,

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad |z| < 1.$$

A function $f \in \mathcal{M}$ is called *convex* if f is univalent analytic and the image $f(\mathbb{D})$ is convex, in other words,

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad |z| < 1.$$

For a constant $\alpha \in [0,1)$, a function $f \in \mathcal{M}$ is called *starlike of order* α if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \quad |z| < 1.$$

Strohhäcker theorem (cf. [3]) asserts that *a convex function is starlike of order* $1/2$. Starlike functions of order $1/2$ play an important role also in the theory of convolution (Hadamard product) (cf. [9]).

Note that these properties are not preserved by post-composition of Möbius maps unlike univalence. Therefore, we do need the third normalization $f''(0) = 2c$.

4. Main results

Theorem 4.1 (Starlikeness Theorem)**.** *Let A be a weight function and c be a complex number. Suppose*

$$2\int_0^1 {U_0}'(x)U_1(x)dx + |c|U_1(1)^2 \le 1.$$

If a function $f \in \mathcal{M}(c)$ satisfies $|S_f(z)| \le A(|z|)$ in $|z| < 1$, then f is starlike of order $1/2$.

As the special case when A is a positive constant and $c = 0$, we obtain

Corollary 4.2. *Let $C_0 = 2\beta_0^2 \approx 2.37036$, where β_0 is the unique positive root of the equation* $\sinh(2\beta) = 4\beta$. *If a function $f \in \mathcal{M}(0)$ satisfies the inequality $|S_f(z)| \le C_0$ in $|z| < 1$, then f is a starlike function of order* $1/2$. *The constant C_0 is sharp.*

Gabriel [4] proved that $|S_f(z)| \le C_0'$ implies starlikeness of $f \in \mathcal{M}(0)$, where $C_0' = 2{\beta_0'}^2 \approx 2.71707$ and β_0' is the unique root of the equation $2\beta = \tan\beta$ in $0 < \beta < \pi/2$.

On the other hand, Chiang [2] showed that C_0' cannot be replaced by a larger number than $C_0'' = (\xi^2 + \eta^2)/2 \approx 4.6351$, where ξ and η are the smallest positive roots of the equations $\xi \tan\xi = -1$ and $\eta \tanh\eta = 1$. By some experiments, it is likely that C_0'' is the best possible constant for starlikeness.

Theorem 4.3 (Convexity Theorem)**.** *Let A be a weight function and c be a complex number. Suppose that the functions V_0 and V_1 satisfy the inequalities*

$$V_0(x) - |c|V_1(x) > 0, \ \ 0 \le x < 1,$$

and

$$-\lim_{x\to 1-}\frac{V_0{}'(x)-|c|V_1{}'(x)}{V_0(x)-|c|V_1(x)}\le\frac{1}{2}. \tag{4.1}$$

If a function $f\in\mathcal{M}(c)$ *satisfies* $|S_f(z)|\le A(|z|)$ *in* $|z|<1$, *then* f *is convex.*

As a corollary, we obtain an improvement of a result of Chiang [2].

Corollary 4.4. *Let* $C_1=2\beta_1^2\approx 0.853526$, *where* β_1 *is the unique root of the equation* $2\beta\tan\beta=1$ *in* $0<\beta<\pi/2$. *If a function* $f\in\mathcal{M}(0)$ *satisfies the inequality* $|S_f(z)|\le C_1$ *in* $|z|<1$, *then* f *is a convex function.*

The constant C_1 above is not sharp. More precisely, C_1 is the sharp constant for which $|S_f(z)|\le C_1$ implies the inequality $|zf''(z)/f'(z)|<1$ in $|z|<1$. As Chiang [2] showed, the constant C_1 cannot be replaced by a larger number than $C_1'=2{\beta_1'}^2\approx 1.19105$, where β_1' is the unique positive root of the equation $\beta\tanh\beta=1/2$.

5. Growth theorems

Our main theorems are based on some growth theorems for solutions to the ODE introduced earlier.

Lemma 5.1. *Let* A *be a weight function and suppose that* $|\varphi(z)|\le A(|z|)$ *in* $|z|<1$. *The solutions* y_0 *and* y_1 *to the differential equation* $2y''+\varphi y=0$ *in* $\mathbb{D}$ *with the initial conditions* $y_0(0)=1, y_0'(0)=0, y_1(0)=0, y_1'(0)=1$ *then satisfy the inequalities*

$$\begin{aligned}\tilde{V_0}(|z|,A)\le&|y_0(z)|\le U_0(|z|,A),\\ &|y_0'(z)|\le U_0{}'(|z|,A),\\ \tilde{V_1}(|z|,A)\le&|y_1(z)|\le U_1(|z|,A),\\ &|y_1'(z)|\le U_1{}'(|z|,A)\end{aligned}$$

for $z\in\mathbb{D}$, *where* $\tilde{V}(x)=V(x)$ *for* $0\le x<x_0$ *and* $\tilde{V}(x)=0$ *for* $x\ge x_0$ *and* x_0 *is the smallest positive zero of* $V(x)$ (*if there is no such zero, set* $x_0=1$).

Lemma 5.2. *Under the same hypothesis as in the previous lemma, let* $y_2=y_0-cy_1$, *where* c *is a complex constant for which the function* $V_2=V_0-|c|V_1$ *is positive on* $(0,1)$. *Then the inequality*

$$\left|\frac{y_2'(z)}{y_2(z)}\right|\le-\frac{V_2{}'(|z|)}{V_2(|z|)} \tag{5.1}$$

holds for every $z\in\mathbb{D}$.

Idea of proof. For a fixed $\zeta\in\partial\mathbb{D}$, we set $w(t)=y_2'(t\zeta)/y_2(t\zeta)$ and $v(t)=-V_2'(t)/V_2(t)$. Then, the function w satisfies the Riccati equation

$$w'=-\frac{\varphi}{2}-w^2.$$

Hence, the function $u(t)=|w(t)|$ satisfies the differential inequality

$$u'\le|w'|\le\frac{A}{2}+u^2.$$

Similarly, the function v satisfies $v' = A/2 + v^2$. The following is a specialized comparison theorem for the present situation, from which the desired inequality follows.

Lemma 5.3 (cf. Walter [14, p. 96]).
Let A be a non-negative continuous function on $[0,1)$ and set $Pw = w' - A/2 - w^2$. If absolutely continuous real-valued functions u, v on $[0,1)$ satisfy the inequalities
(a) $Pu \le Pv$ a.e. in $[0,1)$ and
(b) $u(0) \le v(0)$,
then $u \le v$ holds in $[0,1)$.

Integrating the inequality (5.1), we obtain the following result as a corollary.

Corollary 5.4. *Under the same circumstances as in Lemma 5.2, the inequality $|\log y_2(z)| \le -\log V_2(|z|)$ holds in $|z| < 1$ and, in particular,*

$$V_0(|z|) - kV_1(|z|) \le |y_0(z) - cy_1(z)| \le \frac{1}{V_0(|z|) - kV_1(|z|)}, \quad |z| < 1.$$

Proof of Starlikeness Theorem. Let $f = y_1/y_2$, where $y_2 = y_0 - cy_1$. Then the quantity $p(z) = zf'(z)/f(z)$ satisfies

$$\begin{aligned}\frac{1}{p(z)} = \frac{y_1(z)y_2(z)}{z} &= \int_0^1 \big(y_1y_2\big)'(tz)dt \\ &= 1 + 2\int_0^1 y_1(tz)y_2'(tz)dt.\end{aligned}$$

We use the growth theorem to get

$$\begin{aligned}\left|\frac{1}{p(z)} - 1\right| &\le 2\int_0^1 U_1(t|z|){U_2}'(t|z|)dt \\ &\le 2\int_0^1 U_1(t){U_2}'(t)dt \\ &= 2\int_0^1 U_1(t){U_0}'(t)dt + |c|U_1(1)^2.\end{aligned}$$

We now conclude that $|1/p(z) - 1| < 1$, which is equivalent to $\operatorname{Re} p(z) > 1/2$.

Proof of Convexity Theorem. Use the same notation as in the previous proof. Further we set $V_2 = V_0 - |c|V_1$. Then, since $f' = y_2^{-2}$,

$$1 + \frac{zf''(z)}{f'(z)} = 1 - 2z\,\frac{y_2'(z)}{y_2(z)}.$$

By the second growth lemma,

$$\left|2z\,\frac{y_2'(z)}{y_2(z)}\right| \le -2|z|\,\frac{{V_2}'(|z|)}{V_2(|z|)}.$$

The last term is certainly not greater than 1. Therefore, $\operatorname{Re}\left(1 + zf''(z)/f'(z)\right) > 0$.

6. Examples

The simplest case is when A is a positive constant. If we write $A = 2\beta^2$, where β is a positive number, then

$$\begin{aligned} U_0(x) &= \cosh(\beta x), \\ U_1(x) &= \sinh(\beta x)/\beta, \\ V_0(x) &= \cos(\beta x), \\ V_1(x) &= \sin(\beta x)/\beta. \end{aligned}$$

For $A(x) = C(1-x^2)^{-2}$, where the constant C is allowed to be negative for convenience. If we write $C = 2(4\alpha^2 - 1)$, then

$$\begin{aligned} U_0(x) &= \sqrt{1-x^2}\cosh\left[\alpha\log\left(\frac{1+x}{1-x}\right)\right], \\ U_1(x) &= \frac{\sqrt{1-x^2}}{2\alpha}\sinh\left[\alpha\log\left(\frac{1+x}{1-x}\right)\right]. \end{aligned}$$

However, in this case, we cannot expect a good result concerning convexity or starlikeness as is described in [6].

For $A(x) = C(1-x^2)^{-1}$ with positive constant $C = (1-\alpha^2)/2$,

$$\begin{aligned} U_0(x) &= F(-\tfrac{1+\alpha}{4}, -\tfrac{1-\alpha}{4}; \tfrac{1}{2}; x^2) \\ U_1(x) &= x\,F(\tfrac{1+\alpha}{4}, \tfrac{1-\alpha}{4}; \tfrac{3}{2}; x^2), \end{aligned}$$

where $F(a, b; c; x)$ stands for the hypergeometric function.

As a corollary, we get

Corollary 6.1. *Let $C_2 = (1+\beta_2^2)/2 \approx 1.52444$, where β_2 is the unique positive root of the equation*

$$\int_0^1 x^2 F(\tfrac{3+i\beta}{4}, \tfrac{3-i\beta}{4}; \tfrac{3}{2}; x^2) F(\tfrac{1+i\beta}{4}, \tfrac{1-i\beta}{4}; \tfrac{3}{2}; x^2)dx = \frac{2}{1+\beta^2}.$$

If a function $f \in \mathcal{M}(0)$ satisfies the inequality $|S_f(z)| \le C_2/(1-|z|^2)$ in $|z| < 1$, then f is a starlike function of order $1/2$. The constant C_2 is sharp.

Note that the convexity counterpart for this choice, unfortunately, does not hold because the left-hand side in (4.1) diverges for any choice of c. Indeed, the exponent -1 is critical. If $0 \le \mu < 1$, then the left-hand side in (4.1) converges for $A(x) = C(1-x^2)^{-\mu}$ (see [6] for details).

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