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On Normal Families of Homeomorphisms of the Spere

(Presented by Academician of the NAS of Ukraine I.V. Skrypnik)

We introduce a modulus method to study the precompactness and the compactness of some families of homeomorphisms of the Riemann sphere.

1. The concept of normality plays a decisive role in geometric function theory, complex dynamics and theory of quasiconformal mappings. Especially, that is indispensable to extremal problems, which very often lead to exsitence theorems. Here, for a locally compact metric space X and a complete metric space Y, a family $\mathcal{F}$ of continuous functions $f : X \to Y$ is said to be *normal* if any sequence of functions in $\mathcal{F}$ has a subsequence which converges uniformly on each compact subsets of X. The family $\mathcal{F}$ is said to be *compact* if it is normal and if any limit function of a sequence in $\mathcal{F}$ again bolongs to $\mathcal{F}$.

Many normality criteria have been known for families of analytic functions since Montel founded the argument of normal families, see [5] for example. In the theory of quasiconformal mappings, it is of basic importance that, for a fixed constant $K \geq 1$, the family of all the K-quasiconformal automorphisms of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ normalized so as to fix the three points $0, 1$ and ∞ forms a compact family. This assertion is, however, no longer true if one removes the restriction on the maximal dilatation K. Indeed, the sequence of n-quasiconformal mappings $f_n(z) = |z|^{-1+1/n}z$ of the unit disk $|z| < 1$ defined by does not form a normal family in any neighbourhood of the origin because $|f_n(z)|$ tends to 1 locally uniformly in $0 < |z| < 1$ whereas $f_n(0) = 0$.

Observe that the image of the annulus $r < |z| < R$ under f has modulus $(1/n)\log(R/r)$, which tends to 0 as $n \to \infty$. We present normality criteria for a class of self-homeomorphisms of $\widehat{\mathbb{C}}$ in terms of moduli of the images of annuli so that the above phenomenon will be excluded. Those results will enable us to deduce exsitence theorems for degenerate Beltrami equations under an appropriate integrability condition on the Beltrami coefficients. Similar approach was taken by Lehto [3] and effectively used by Brakalova and Jenkins [1].

2. Let us recall that a doubly connected domain in the Riemann sphere $\widehat{\mathbb{C}}$ is called a ring domain. The modulus t of a ring domain A is the number such that A is conformally equivalent to the annulus $\{1 < |z| < e^t\}$ and this number will be denoted by $\operatorname{mod} A$.

In what follows, we will use both the Euclidean metric $d(z, w) = |z-w|$ and the spherical metric $d^{\mathrm{s}}(z, w) = |z-w|/\sqrt{(1+|z|^2)(1+|w|^2)}$. The spherical measurement will be indicated by the sufix s. For example, $A(z_0, r, R)$ denotes the annulus $r < |z - z_0| < R$ and $A^{\mathrm{s}}(z_0, r, R)$ denotes the spherical annulus $r < d^{\mathrm{s}}(z, z_0) < R$. Here, we set $A(\infty, r, R) = A(0, 1/R, 1/r)$.

A non-negative function $\rho(z, r, R)$ in $(z, r, R) \in \widehat{\mathbb{C}} \times (0, \infty) \times (0, \infty)$, $r < R$, will be called a *modulus constraint* if $\rho(z_0, r, R) \to \infty$ as $r \to 0$ for each $z_0 \in \widehat{\mathbb{C}}$ and for each $R > 0$. We denote by $\mathcal{H}_\rho$ the family of all normalized homeomorphisms $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that the condition

$$\operatorname{mod} f(A(z_0, r, R)) \geq \rho(z_0, r, R)$$

holds for all $z_0 \in \widehat{\mathbb{C}}$ and $r, R \in (0, +\infty)$ with $r < R$.

Similarly, a non-negative function $\rho^{\mathrm{s}}(z, r, R)$ in $(z, r, R) \in \widehat{\mathbb{C}} \times (0, 1) \times (0, 1)$, $r < R$, will be called a *spherical modulus constraint* if $\rho^{\mathrm{s}}(z_0, r, R) \to +\infty$ as $r \to 0$ for each $z_0 \in \widehat{\mathbb{C}}$ and for each $R \in (0, 1)$. We let $\mathcal{H}^{\mathrm{s}}$ be the family of all normalized self-homeomorphisms f such that $\operatorname{mod} f(A^{\mathrm{s}}(z_0, r, R)) \geq \rho^{\mathrm{s}}(z_0, r, R)$ holds for all $0 < r < R < 1$ and $z_0 \in \widehat{\mathbb{C}}$.

The following statements are similar to a result in [3], whose assumption involves the growth of the images of any annuli in $\widehat{\mathbb{C}}$.

Theorem 1. *Let ρ^{s} be a spherical modulus constraint. Then*

1) *${\mathcal{H}^{\mathrm{s}}}_{\rho^{\mathrm{s}}}$ is a compact family with respect to the uniform convergence in $\widehat{\mathbb{C}}$, and*

2) *every $f \in {\mathcal{H}^{\mathrm{s}}}_{\rho^{\mathrm{s}}}$ satisfies the inequality*

$$d^{\mathrm{s}}(f(z_1),\ f(z_2)) \leq Ce^{-\frac{1}{2}\rho^{\mathrm{s}}(z_0, r_1, r_2)}, \quad z_1, z_2 \in B^{\mathrm{s}}(z_0, r_1),$$

for $z_0 \in \widehat{\mathbb{C}}$ and $0 < r_1 < r_2 < 1/2\sqrt{2}$, where $B^{\mathrm{s}}(z, r)$ stands for the spherical disk centered at z with radius r and C is an absolute constant.

The exponent $1/2$ in the above estimate is improved to be 1 if one concentrates on annuli in $\mathbb{C}$.

Theorem 2. *Let ρ be a modulus constraint. Then*

1) *$\mathcal{H}_\rho$ is a compact family with respect to the uniform convergence in $\widehat{\mathbb{C}}$, and*

2) *for each $R > 0$ there is a constant $C = C(R, \rho) > 0$ depending only on R and ρ such that every $f \in \mathcal{H}_\rho$ satisfies*

$$|f(z_1) - f(z_2)| \leq Ce^{-\rho(z_0, r_1, r_2)}, \quad z_1, z_2 \in B(z_0, r_1),$$

for $|z_0| \leq R$ and $0 < r_1 < r_2 < R$, where $B(z, r)$ is the disk centered at z with radius r.

The proof of these theorems are based on the following auxiliary lemmas, which may be of independent interest.

Lemma 1. *Let B be an arbitrary ring domain in $\widehat{\mathbb{C}}$ and let E_0 and E_1 be the components of $\widehat{\mathbb{C}} \setminus B$. Then the inequality*

$$\min\{\operatorname{diam}^{\mathrm{s}} E_0, \operatorname{diam}^{\mathrm{s}} E_1\} \leq C_1 e^{-\frac{1}{2}\operatorname{mod} B}$$

holds where C_1 is an absolute constant.

Indeed, one can take $C_1 = 2.56957\ldots$, see [2].

Lemma 2. *Let B be a ring domain in $\mathbb{C}$ whose complement in $\widehat{\mathbb{C}}$ consists of the bounded component E_0 and the unbounded component E_1. Then there are absolute constants $C_2 > 0$ and $C_3 < \infty$ such that*

$$\operatorname{diam} E_0 \leq C_3 \operatorname{dist}(E_0, E_1) e^{-\operatorname{mod} B},$$

provided that $\operatorname{mod} B > C_2$.

Actually, the above assertion holds for $C_2 = 5\log 2$ and $C_3 = 64$, see [2].

3. The above theorems can be applied to the study of the Beltrami equation

$$f_{\bar{z}} = \mu(z)\, f_z \quad \text{a.e.}$$

when the measurable coefficient μ satisfies the conditions $|\mu(z)| < 1$ a.e. and $\|\mu\|_\infty = 1$.

Given μ, let us introduce the following approximation procedure. For $n = 1, 2, ...$, we set $\mu_n(z) = \mu(z)$ if $|\mu(z)| \leq 1 - 1/n$, and $\mu_n(z) = 0$ otherwise, and denote by f_n the sequence of normalized quasiconformal automorphisms of $\widehat{\mathbb{C}}$ having μ_n as its complex dilatation. The existence of such f_n is guaranteed by the measurable Riemann mapping theorem, see [4], p. 183. We will call such f_n the *canonical approximating sequence* for μ. The topological structure of the family $\{f_n\}$ with respect to the locally uniform convergence under some additional assumptions on μ is described in Theorem 3 below. Before the statement, we give some definitions.

Let $\mu \in L^\infty(\Omega)$ be a Beltrami coefficient with $|\mu| < 1$ a.e. We set

$$D_{\mu,z_0}(z) = \frac{\left|1 - \mu(z)\frac{\bar{z}-\bar{z}_0}{z-z_0}\right|^2}{1 - |\mu(z)|^2}$$

if $z_0 \in \mathbb{C}$, and set $D_{\mu,\infty}(z) = D_{\mu,0}(z)$.

Let $\mathcal{H}$ be the set of functions $H : [0, +\infty) \to \mathbb{R}$ satisfying the three conditions: 1) $H(x)$ is continuous and strictly increasing in $x \in [x_0, +\infty)$ and $H(x) = H(x_0)$ for $x \in [0, x_0]$ for some $x_0 \geq 0$; 2) The function $e^{H(x)}$ is convex in $x \in [0, +\infty)$; and 3)

$$\int_1^{+\infty} \frac{H(x)dx}{x^2} = +\infty.$$

Theorem 3 *Let μ be a Beltrami coefficient in $\mathbb{C}$. Assume that for each $z_0 \in \widehat{\mathbb{C}}$ there is a function $H = H_{z_0}$ in $\mathcal{H}$ such that for some positive constants $M = M(z_0)$ and $r_0 = r_0(z_0)$ the condition*

$$\int_{B(z_0,r_0)} e^{H(D_{\mu,z_0})} dm \leq M$$

holds for $z_0 \in \mathbb{C}$, while the above condition is replaced by

$$\int_{B(\infty,r_0)} e^{H(D_{\mu,0}(z))} \frac{dm(z)}{|z|^4} \leq M$$

if $z_0 = \infty$. Then the canonical approximating sequence f_n for μ forms a normal family with respect to the uniform convergence in $\widehat{\mathbb{C}}$ and every limit function f of this sequence is a self-homeomorphism of $\widehat{\mathbb{C}}$. Moreover, f admits the following modulus of continuity estimate at each point z_0 with $|z_0| \leq R_0$, where R_0 is an arbitrary positive number:

$$|f(z) - f(z_0)| \leq C \exp\left\{-\int_{1+c}^{2m+c} \frac{dt}{2H^{-1}(t)}\right\}$$

for $|z - z_0| < r_1$ and $0 < r_1 \leq \min\{r_0, R_0\}$, where $m = \log(r_1/|z - z_0|)$, $c = \log(M/\pi r_1^2)$ and C is a constant depending only on μ and R_0.

The proof of the theorem is based on Theorem 1 and the estimate on the moduli of ring domains under homeomorphisms in the Sobolev space $W^{1,1}_{\mathrm{loc}}$ stated in the lemma below.

Lemma 3. *Let $f : A \to \mathbb{C}$ be an injective continuous map in $W^{1,1}_{\mathrm{loc}}(A)$ for $A = A(z_0, r_0e^{-t}, r_0)$ which satisfies $f_{\bar{z}} = \mu f_z$ in A for $\mu \in L^\infty(A)$ with $\|\mu\|_\infty \leq 1$. Suppose that a function $H \in \mathcal{H}$ satisfies*

$$\int_A e^{H(D_{\mu,z_0}(z))} dm(z) \leq M, \quad \text{if } z_0 \in \mathbb{C}, \text{ and}$$

$$\int_A e^{H(D_{\mu,0}(z))} \frac{dm(z)}{|z|^4} \leq M, \quad \text{if } z_0 = \infty.$$

Then one has

$$\operatorname{mod} f(A(z_0, r_0e^{-t}, r_0)) \geq \int_{1/2}^{t} \frac{dt}{H^{-1}(2t + \log(M/\pi r_0^2))}$$

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